Homework

You will need some identities for fermion operators in the first 3 problems:

$\left\{c_{\mathbf{k}\sigma}, c^{+}_{\mathbf{k}'\sigma'}\right\} = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}$, $\left\{c_{\mathbf{k}\sigma}, c_{\mathbf{k}'\sigma'}\right\} = \left\{c^{+}_{\mathbf{k}\sigma}, c^{+}_{\mathbf{k}'\sigma'}\right\} = 0$, and $c_{\mathbf{k}\sigma}\left|0\right\rangle = \mathbf{0}$.

1. **Particle number fluctuation**: Prove that the particle number fluctuation of the BCS ground state $\left|\Psi_G\right\rangle = \prod_{\mathbf{k}}\left(u_{\mathbf{k}} + v_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow} c_{-\mathbf{k}\downarrow}\right)\left|0\right\rangle$ is $\left\langle N^2\right\rangle - \left\langle N\right\rangle^2 = 4\sum_{\mathbf{k}} u_{\mathbf{k}}^2 v_{\mathbf{k}}^2$. The particle number operator is $N = \sum_{\mathbf{k}\sigma} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}$. (Note that $\left|u_{\mathbf{k}}\right|^2 + \left|v_{\mathbf{k}}\right|^2 = 1$)

2. **The quasi-particle operators**: Show that $\gamma^{+}_{\mathbf{k}0}\left|\Psi_G\right\rangle = c^{+}_{\mathbf{k}\uparrow}\prod_{\mathbf{l}\neq\mathbf{k}}\left(u_{\mathbf{l}} + v_{\mathbf{l}} c^{+}_{\mathbf{l}\uparrow} c_{-\mathbf{l}\downarrow}\right)\left|0\right\rangle$ and $\gamma^{+}_{\mathbf{k}1}\left|\Psi_G\right\rangle = c^{+}_{-\mathbf{k}\downarrow}\prod_{\mathbf{l}\neq\mathbf{k}}\left(u_{\mathbf{l}} + v_{\mathbf{l}} c^{+}_{\mathbf{l}\uparrow} c_{-\mathbf{l}\downarrow}\right)\left|0\right\rangle$. How do you interpret these results?

3. **Variational method**: Show that the energy expectation value for the BCS ground state $\left|\Psi_G\right\rangle$ in the pairing hamiltonian $H = 2\sum_{\mathbf{k}\sigma}\varepsilon_{\mathbf{k}\sigma} n^2_{\mathbf{k}\sigma} + \sum_{\mathbf{kl}} V_{\mathbf{kl}} c^{+}_{\mathbf{k}\uparrow} c^{+}_{-\mathbf{k}\downarrow} c_{-\mathbf{l}\downarrow} c_{\mathbf{l}\uparrow}$ is $\left\langle\Psi_G\right| H - \mu N \left|\Psi_G\right\rangle = 2\sum_{\mathbf{k}}\xi_{\mathbf{k}} v^2_{\mathbf{k}} + \sum_{\mathbf{kl}} V_{\mathbf{kl}} u_{\mathbf{k}} v_{\mathbf{k}} u_{\mathbf{l}} v_{\mathbf{l}}$. Parameterizing $u_{\mathbf{k}} = \sin\theta_{\mathbf{k}}$ and $v_{\mathbf{k}} = \cos\theta_{\mathbf{k}}$ and minimizing the energy with respect to $\theta_{\mathbf{k}}$, one gets a self-consistent equation $\tan 2\theta_{\mathbf{k}} = \dfrac{\sum_{\mathbf{kl}} V_{\mathbf{kl}}\sin 2\theta_{\mathbf{l}}}{2\xi_{\mathbf{k}}}$. Compare this result with that derived by directly diagonizing the hamiltonian.

4. **Temperature dependence of the gap**: Numerically solve the gap equation in thermal equilibrium $\dfrac{1}{N(0)V} = \int_0^{\hbar\omega_c}\dfrac{\tanh\left(\beta E/2\right)}{E}d\xi$ in which $E = \sqrt{\xi^2 + \Delta(T)^2}$. Plot the temperature dependence of the gap. (You can assume a big $\hbar\omega_c$ value, that is, $\beta\hbar\omega_c \gg 1$ in the integral.)

5. **The paramagnetic limit:** You can use the same numerical method to calculate the critical field H_p in the paramagnetic limit in which the electron spins are polarized by the external field. In this scenario, the particle distribution function is changed to be a non-equilibrium distribution, $f'(E) = \dfrac{1}{2} f\left(E - \mu_{\uparrow}\right) + \dfrac{1}{2} f\left(E - \mu_{\downarrow}\right)$ in which *f*(*E*) is the Fermi-Dirac distribution. $\mu_{\uparrow}$ and $\mu_{\downarrow}$ are the spin chemical potentials that $\mu_{\uparrow} = \mu - \dfrac{1}{2} g\mu_B H$ and $\mu_{\downarrow} = \mu + \dfrac{1}{2} g\mu_B H$. To calculate the critical field, apply the gap equation in non-equilibrium conditions $\dfrac{1}{N(0)V} = \int_0^{\hbar\omega_c}\dfrac{1 - 2f'(E)}{E}d\xi$. H_p is defined as the field at which $\Delta = 0$. What is the temperature dependence of H_p?