11. Tunneling and Josephson Effect

Density of states

$\gamma_{k0}\left|\Psi_G\right\rangle = \gamma_{k1}\left|\Psi_G\right\rangle = 0$ $\qquad$ $\left|\Psi_G\right\rangle$ is vacuum for γ-particles

$$\gamma_{k0}^{+}\left|\Psi_G\right\rangle = c_{k\uparrow}^{+}\prod_{l\neq k}\left(u_l + v_l c_{l\uparrow}^{+} c_{-l\downarrow}^{+}\right)\left|0\right\rangle = 0$$

$$\gamma_{k1}^{+}\left|\Psi_G\right\rangle = c_{-k\downarrow}^{+}\prod_{l\neq k}\left(u_l + v_l c_{l\uparrow}^{?} c_{-l\downarrow}\right)\left|0\right\rangle = 0$$

$\gamma_k \leftrightarrow c_k$ $\qquad$ One-to-one map

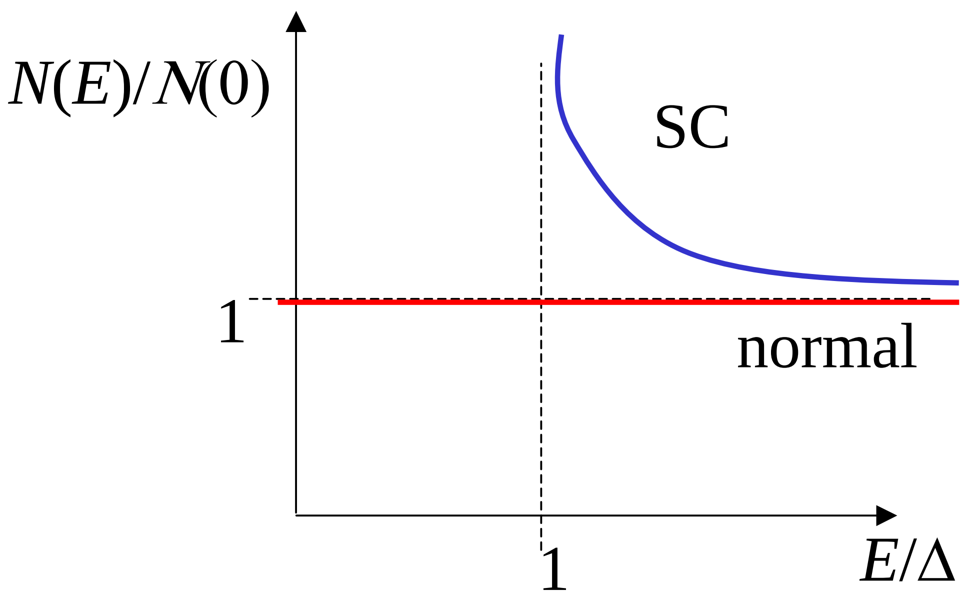
$$N_S\left(E\right)dE = N_N\left(\xi\right)d\xi$$

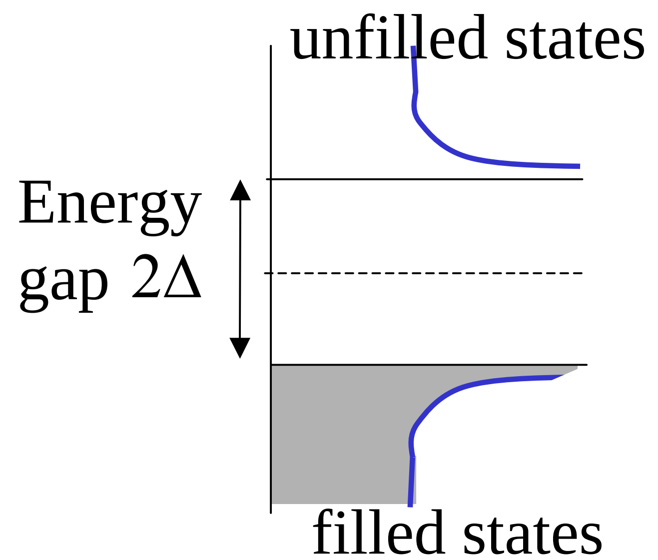
Number of states are equal

$$\frac{N_S\left(E\right)}{N_N\left(\xi\right)} = \frac{d\xi}{dE} = \begin{cases} \dfrac{E}{\sqrt{E^2-\Delta^2}} & \text{for } E>0 \\ 0 & \text{for } E<0 \end{cases}$$

Semiconductor model

$N_N(\xi)$ is the normal density of state at Fermi level





Normal-normal tunneling

$$I_{1\to 2} = A|T|^2 \int_{-\infty}^{\infty} \underbrace{N_1(E)\, f(E)}_{\text{filled states}}\, \underbrace{N_2(E+eV)\left[1 - f(E+eV)\right]}_{\text{unoccupied states}} dE$$

net current

$$I_{NN} = A|T|^2 \int_{-\infty}^{\infty} N_1(E)\, N_2(E+eV)\left[f(E) - f(E+eV)\right] dE$$

$$I_{NN} = A|T|^2 N_1(0)\, N_2(0) \int_{-\infty}^{\infty} \left[f(E) - f(E+eV)\right] dE$$

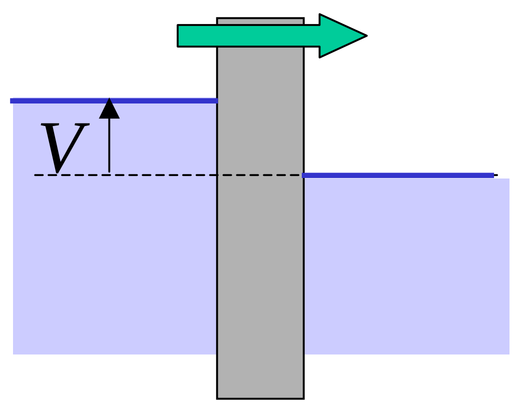
$$= A|T|^2 N_1(0)\, N_2(0)\, eV$$

$$\equiv G_{NN} V$$

For normal metal,
DOS is constant of E

normal state conductance

net current



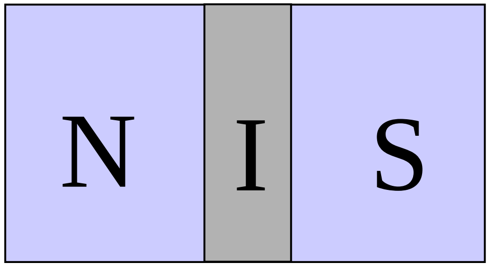
$$f(E) = \frac{1}{1+e^{\beta E}}$$ Fermi-Dirac distribution

Normal-superconductor tunneling

$$I_{NS} = A\left|T\right|^2 N_1(0)\int_{-\infty}^{\infty} N_{2S}(E)\left[f(E) - f(E+eV)\right]dE$$

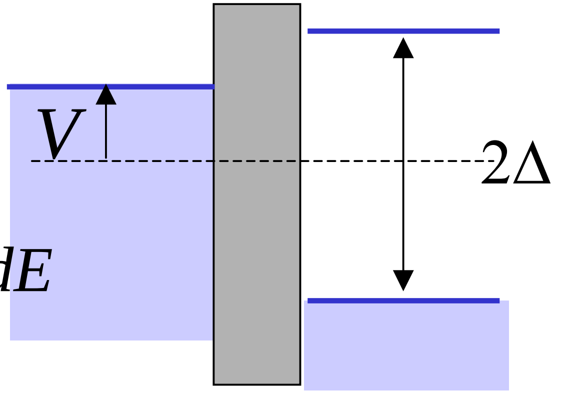
$$= \frac{G_{NN}}{e}\int_{-\infty}^{\infty} \frac{N_{2S}(E)}{N_2(0)}\left[f(E) - f(E+eV)\right]dE$$

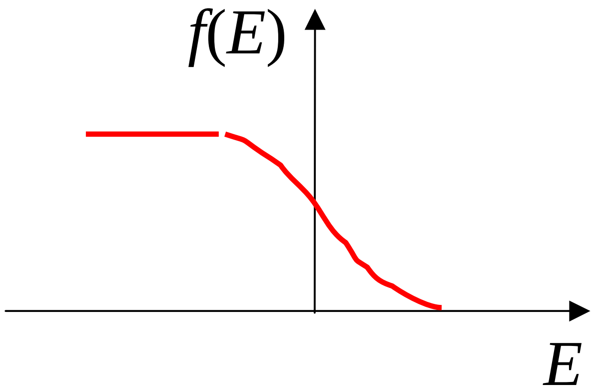
$$= \frac{G_{NN}}{e}\int_{-\infty}^{\infty} \frac{E}{\sqrt{E^2-\Delta^2}}\left[f(E) - f(E+eV)\right]dE$$

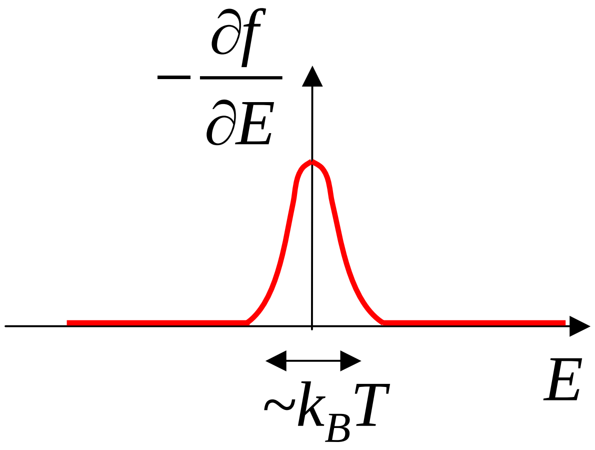


for $E>\Delta$ and $E<-\Delta$

$$G_{NS} = \frac{dI_{NS}}{dV} = G_{NN}\int_{-\infty}^{\infty} \frac{E}{\sqrt{E^2-\Delta^2}}\left[-\frac{\partial f(E+eV)}{\partial E}\right]dE$$





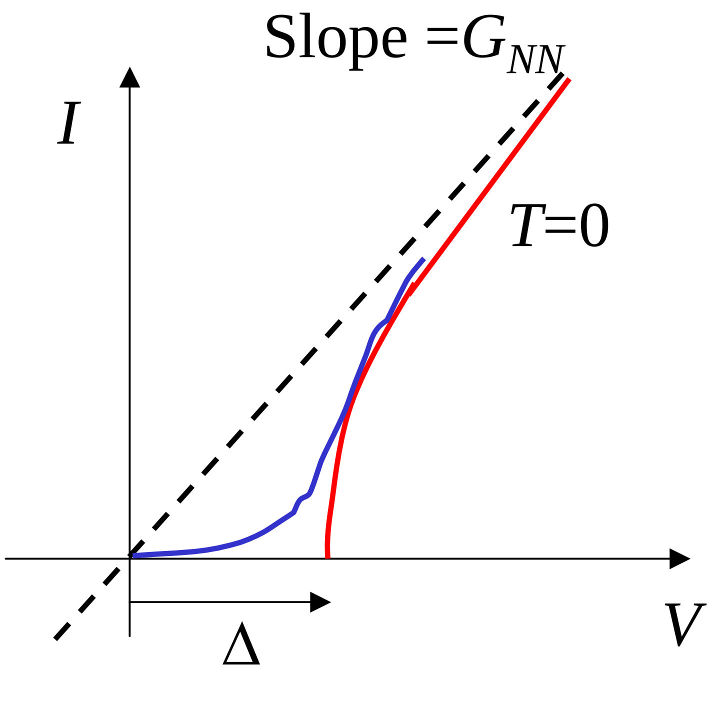


At T=0, $f(E)$ is a step function, and

$$-\frac{\partial f}{\partial E}$$ is a δ-function

$$G_{NS}\left(T \to 0\right) = G_{NN} \frac{E}{\sqrt{E^2 - \Delta^2}}$$

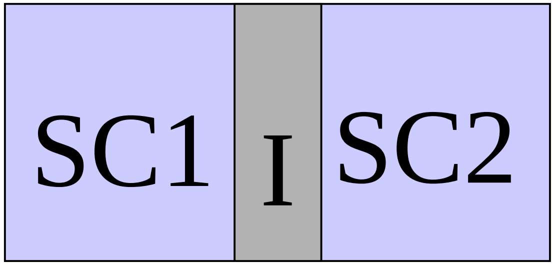
A direct measurement of energy gap and DOS



SIS junction

The superconducting leads can be expressed with a macroscopic wave function

Ψ_1 and Ψ_2



Wave function Ψ_1 Ψ_2

The wave equations

Eigenenergies for SC1 and SC2

$$i\hbar \frac{\partial \Psi_1}{\partial t} = U_1 \Psi_1 + K \Psi_2$$

$$i\hbar \frac{\partial \Psi_2}{\partial t} = U_2 \Psi_2 + K \Psi_1$$

K is the coupling energy between the two wave functions

$$\Psi_1 = \left|\Psi_1\right| e^{i\theta_1} \qquad \Psi_2 = \left|\Psi_2\right| e^{i\theta_2}$$

$$i\hbar\left(\frac{\partial\left|\Psi_1\right|}{\partial t} + i\left|\Psi_1\right|\frac{\partial\theta_1}{\partial t}\right) = U_1\left|\Psi_1\right| + K\left|\Psi_2\right|e^{i(\theta_2-\theta_1)}$$

$$i\hbar\left(\frac{\partial\left|\Psi_2\right|}{\partial t} + i\left|\Psi_2\right|\frac{\partial\theta_2}{\partial t}\right) = U_2\left|\Psi_2\right| + K\left|\Psi_1\right|e^{i(\theta_1-\theta_2)}$$

Real part: $$-\hbar\left|\Psi_1\right|\frac{\partial\theta_1}{\partial t} = U_1\left|\Psi_1\right| + K\left|\Psi_2\right|\cos\left(\theta_2-\theta_1\right)$$

Phase change

Imaginary part: $$\hbar\frac{\partial\left|\Psi_1\right|}{\partial t} = K\left|\Psi_2\right|\sin\left(\theta_2-\theta_1\right)$$

Particle number change

Josephson effect

$$-\frac{\partial \theta_1}{\partial t} = \frac{U_1}{\hbar} + \frac{K}{\hbar}\frac{|\Psi_2|}{|\Psi_1|}\cos(\theta_2 - \theta_1)$$

with $|\Psi_2| \approx |\Psi_1|$

$$-\frac{\partial \theta_2}{\partial t} = \frac{U_2}{\hbar} + \frac{K}{\hbar}\frac{|\Psi_1|}{|\Psi_2|}\cos(\theta_1 - \theta_2)$$

$\phi = \theta_1 - \theta_2$

Phase difference $$\frac{\partial \phi}{\partial t} = \frac{\partial \theta_1}{\partial t} - \frac{\partial \theta_2}{\partial t} = \frac{U_2 - U_1}{\hbar} = \frac{2eV}{\hbar}$$

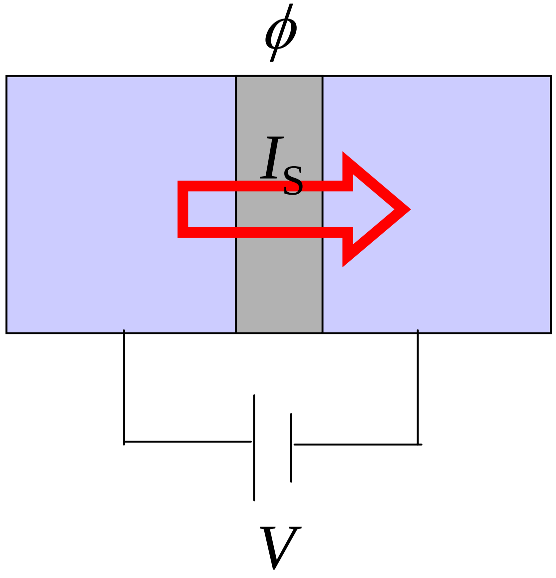
(e*=-2e)

$$\frac{\partial |\Psi_1|^2}{\partial t} = 2|\Psi_1|\frac{\partial |\Psi_1|}{\partial t} = \frac{2K}{\hbar}|\Psi_1||\Psi_2|\sin(\theta_2 - \theta_1)$$

$$\frac{\partial |\Psi_2|^2}{\partial t} = 2|\Psi_2|\frac{\partial |\Psi_2|}{\partial t} = -\frac{2K}{\hbar}|\Psi_1||\Psi_2|\sin(\theta_2 - \theta_1)$$

Current= $$I_S = -e^*\frac{\partial |\Psi_1|^2}{\partial t} = -\frac{2e^*K}{\hbar}|\Psi_1||\Psi_2|\sin\phi \equiv I_C \sin\phi$$

Josephson energy



Starting from zero phase difference, the voltage source drives the junction to a phase difference= ϕ

The electrical power delivered= $I_S V$

Energy stored in juntion

$$= \int I_S V dt = \int I_S \frac{\hbar}{2e} \frac{\partial \phi}{\partial t} dt = \frac{\hbar}{2e} \int I_S d\phi = \frac{\hbar}{2e} I_C \int \sin\phi \, d\phi$$

$$= -E_J \cos\phi + \text{constant}$$

$$E_J = \frac{2e}{\hbar} I_C$$ The Josephson coupling energy

ac Josephson effect

$$I_S = I_C \sin\phi$$

$$\frac{\partial\phi}{\partial t} = \frac{2eV}{\hbar}$$

For V=0, ϕ=constant, current is constant of time

For $V \neq 0$, $\phi = \omega_0 t$

$$\omega_0 = \frac{2eV_0}{\hbar}$$

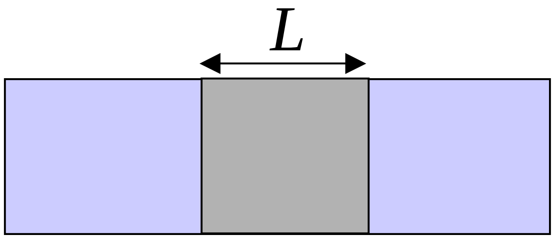
$$\Rightarrow \quad I_S = I_C \sin\left(\omega_0 t + \phi(0)\right)$$

At finite DC voltage, the supercurrent is alternating with a angular frequency $\omega_0 = 2eV_0/\hbar$

Ginzburg-Landau theory for 1D weak link

Apply the G-L eqns in 1D

$$\xi^2 \frac{d^2 f}{dx^2} + f - f^3 = 0$$



When the junction length is small, $\frac{\xi}{L} \gg 1$

we have Poisson eqn: $\frac{d^2 f}{dx^2} \simeq 0$

General solution: $f = a + bx$

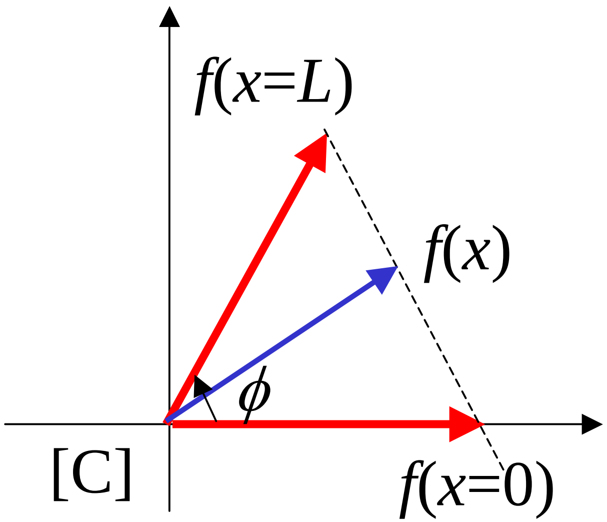
Impose boundary conditions

$$\Rightarrow \quad f = \left(1 - \frac{x}{L}\right) + \left(\frac{x}{L}\right) e^{i\phi}$$

Boundary conditions

x=0, phase=0 $\quad |f| = 1$

x=L, phase = ϕ $\quad |f| = 1$



Supercurrent:

$$J \propto \frac{1}{i}\left(f^{*}\nabla f - f\nabla f^{*} \right)$$

$$= \frac{2}{L}\,\mathrm{Im}\left(\left(\left(1-\frac{x}{L}\right)+\left(\frac{x}{L}\right)e^{-i\phi}\right)\left(-1+e^{i\phi}\right)\right).$$

$$= \frac{2}{L}\sin\phi$$

$\Rightarrow$ $J \propto \sin\phi$ $\qquad I_S = I_C \sin\phi$

Plug in the prefactors

$$I_C = \frac{2e\hbar\Psi_{\infty}^{2}}{m^{*}}\frac{A}{L}$$

A → Junction cross sectional area

L → Junction length

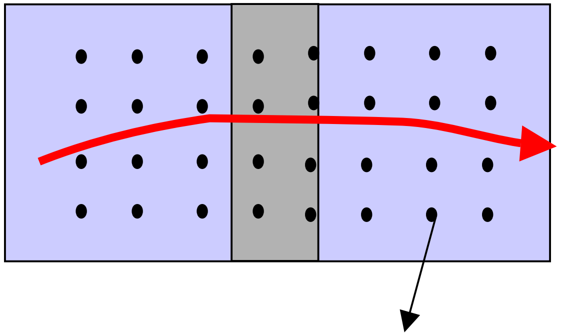
Gauge-invariant phase

The magnetic field affects the superconducting wave function

Consider the total phase change through the red path:

$$\phi = \gamma + \frac{2\pi}{\Phi_0}\int \vec{\mathbf{A}}\cdot d\vec{\mathbf{s}}$$

Gauge-invariant phase



The supercurrent is an gauge invariant quantity, but phase is not.

$$I_S = I_C \sin\gamma$$

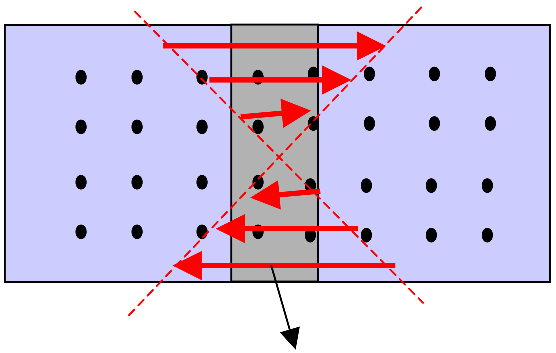
Diffraction pattern in single Josephson junction

Choose the gauge

$$A_x = By$$

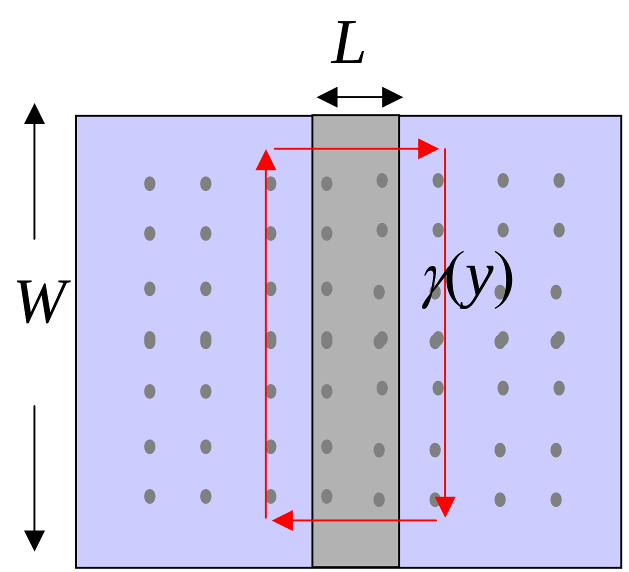
$$A_y = 0$$

$I_S(y)$



Vector potential

$$\frac{2\pi}{\Phi_0}\int \vec{\mathbf{A}}\cdot d\vec{\mathbf{s}} = \frac{2\pi}{\Phi_0}yB\Delta x$$



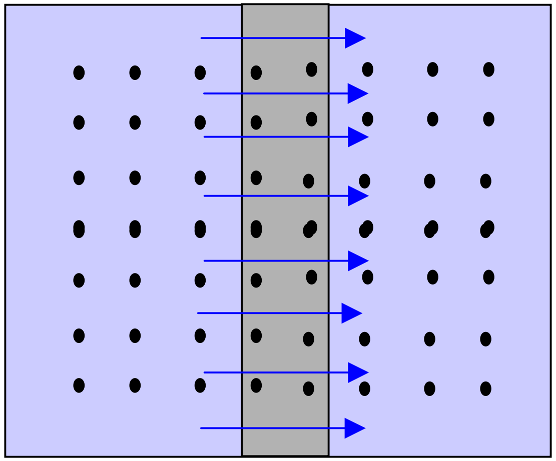
With the single value of superconducting phase, we have the superconducting phase difference are position dependent

$$\gamma(y) = \frac{2\pi}{\Phi_0}yBL$$

The current is a superposition of the difference phase

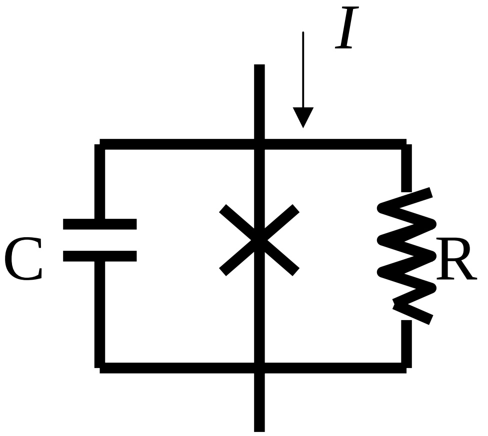
$$I_S = \int dI_S = \frac{I_C}{W}\int_{-W/2}^{W/2} \sin\phi(y)\,dy = \frac{I_C}{W}\int_{-W/2}^{W/2} \sin\left(\frac{2\pi LB}{\Phi_0}y\right)dy$$

→ Single slit diffraction



RCSJ model

Resistively and capacitively shunted junction:



$$I = C\frac{dV}{dt} + \frac{V}{R} + i_C \sin \Delta\phi$$

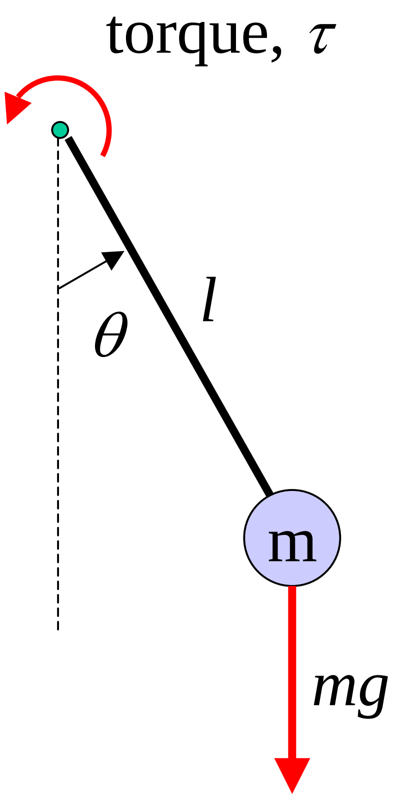
$$V = \frac{\hbar}{2e}\frac{d}{dt}\Delta\phi$$

$$\Rightarrow \quad \frac{C\hbar}{2e}\frac{d^2}{dt^2}\Delta\phi + \frac{\hbar}{2eR}\frac{d}{dt}\Delta\phi + i_C \sin \Delta\phi = I$$

Kinetic term — damping — potential term — Driving force

Pendulum analogy

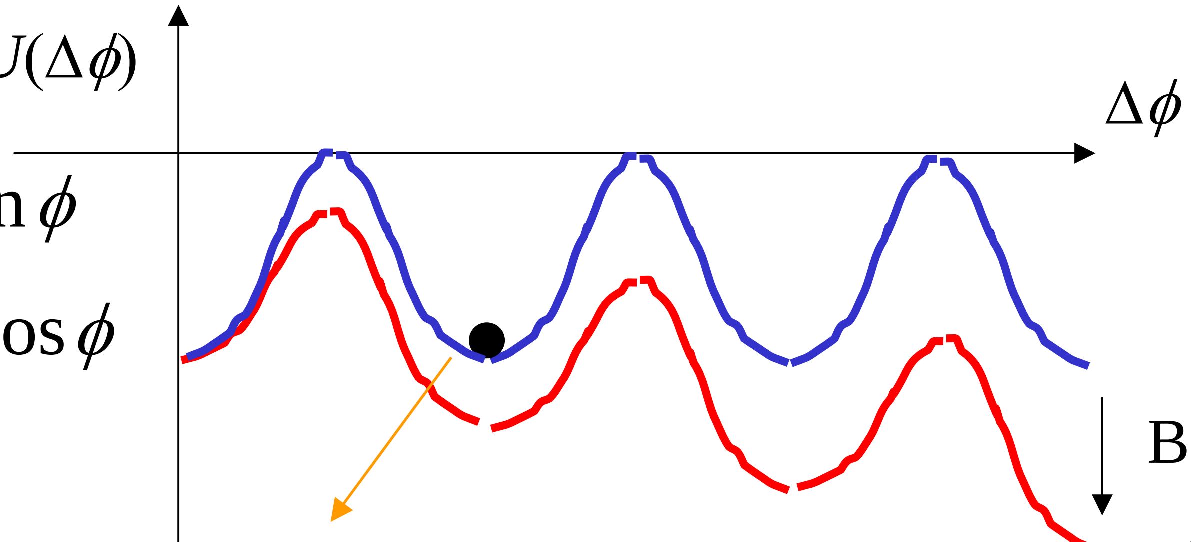
RCSJ model	**pendulum**
$\Delta\phi$	θ
I	τ
C	m
$1/R$	η
i_S	x
V	ω



The classical dynamics of a Josephson junction can be solved

Tilted washboard model

Potential, $U(\Delta\phi)$



$$-\nabla U \propto \sin\phi$$

$$U = -E_J \cos\phi$$

Bias current

Particle mass~ $\left(\frac{\hbar}{2e}\right)^2 C$

$$U = -E_J \cos\phi - \left(\frac{\hbar I}{2e}\right)\Delta\phi$$

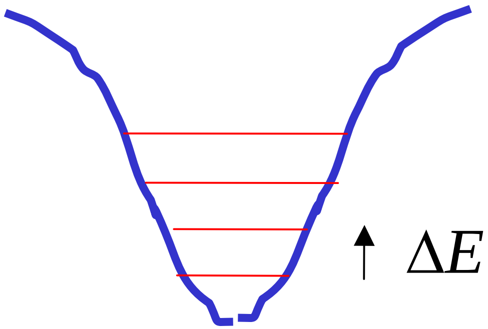
The particle is moving with a friction force proportional to its speed

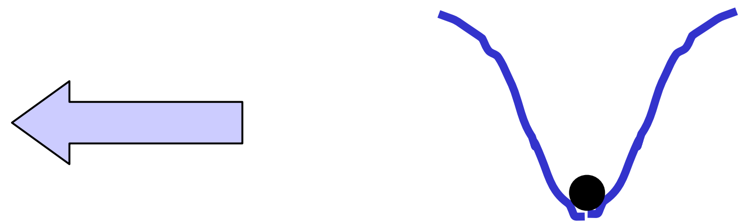
$$\frac{1}{R}\left(\frac{\hbar}{2e}\right)^2 \frac{d}{dt}\Delta\phi$$

Quantum or classical?

In what situation the classical picture is plausible?

Consider the quantum mechanical excitations in the system





1D quantum well

Classical motion around a local potential minimum

$$U = -E_J \cos\phi \approx \boxed{\tfrac{1}{2} E_J \phi^2} - E_J$$

Simple harmonic potential with a spring constant of E_J

Particle mass~ $\left(\frac{\hbar}{2e}\right)^2 C$

$$\Delta E = \hbar\omega = \hbar\sqrt{\frac{\text{spring constant}}{\text{mass}}} = \hbar\sqrt{\frac{E_J}{(\hbar/2e)^2 C}} = 2\sqrt{2E_C E_J}$$

Charging energy $$E_C \equiv \frac{e^2}{2C}$$

If barrier height E_J is much larger than ΔE, the junction can be treated classically.

$$E_J \gg E_C$$

Shapiro steps

Resonance in periodic motion systems

$$V = V_0 + V_1 \cos \omega_1 t$$

DC AC

$$V = \frac{\hbar}{2e}\frac{d}{dt}\Delta\phi \qquad \Delta\phi(t) = \Delta\phi(0) + \boxed{\frac{2eV_0}{\hbar}t} + \frac{2eV_1}{\hbar\omega_1}\sin\omega_1 t$$

constant angular velocity

$$\omega_0 = \frac{2eV_0}{\hbar}$$

The supercurrent

$$I_S = i_C \sin\Delta\phi$$

$$= i_C \sum_n (-1)^n J_n\left(\frac{2eV_1}{\hbar\omega_1}\right)\sin\left(\Delta\phi(0) + \omega_0 t - n\omega_1 t\right)$$

When $\omega_0 = n\omega_1$ there are time independent currents

$$i_C(-1)^n J_n\left(\frac{2eV_1}{\hbar\omega_1}\right)\sin\Delta\phi(0)$$

$$2eV_0 = n\hbar\omega_1 \qquad V_0 = \frac{n\hbar\omega_1}{2e}$$

At the step, the current is

$$-i_C J_n\left(\frac{2eV_1}{\hbar\omega_1}\right) + I_{qp} < \langle I \rangle < i_C J_n\left(\frac{2eV_1}{\hbar\omega_1}\right) + I_{qp}$$

the current has a half width= $i_C J_n\left(\frac{2eV_1}{\hbar\omega_1}\right)$

When irradiated by a microwave, the junction is more likely to be current biased instead of voltage biased