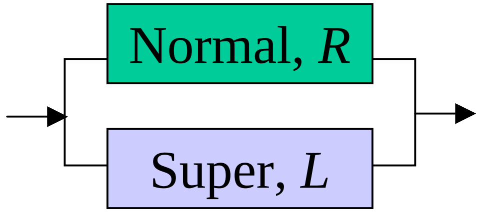
3. Electrodynamics

Two fluid model

Normal electrons and "superelectrons"



Normal electrons have a finite mean free path while superelectrons have infinite mean free path.

In DC field, the current is carried by the superelectrons. There should be no electric field inside the superconductor, otherwise the superelectrons will be accelerated continuously.

In AC field, the superelectrons may lag behind the field and present an *inductive impedance*. In this case the electric field is not necessarily to be zero, and therefore some of the current will carried by the normal electrons.

Typically, L(Henry)~$10^{-12}R_N$(Ohm)

The power dissipated by the normal electrons is negligible.

Result of a perfect conductor

$$J(\omega) = \sigma(\omega) E(\omega)$$

AC conductivity

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau} \qquad \sigma_0 = \frac{ne^2\tau}{m}$$

For superelectons, $\tau \to \infty$

$$= i\frac{ne^2}{m\omega}$$ Inductive impedance

$$-i\omega J(\omega) = \frac{ne^2}{m} E(\omega) \quad \Longrightarrow \quad \frac{\partial J}{\partial t} = \frac{ne^2}{m} E$$

$$\nabla \times E + \frac{\partial B}{\partial t} = 0 \quad \Longrightarrow \quad \frac{\partial}{\partial t}\left(\nabla \times J + \frac{ne^2}{m} B \right) = 0$$

$$\nabla \times B = \mu_0 J + \frac{1}{c^2}\frac{\partial E}{\partial t}$$

$$\frac{\partial}{\partial t}\left(\nabla \times J + \frac{ne^2}{m} B \right) = 0 \quad \Longrightarrow \quad \frac{\partial}{\partial t}\left(\nabla \times \nabla \times B + \mu_0 \frac{ne^2}{m} B \right) = 0$$

use $\nabla \times \nabla \times B = \nabla \left(\nabla \bullet B \right) - \nabla^2 B = -\nabla^2 B$

$$\Longrightarrow \quad \frac{\partial}{\partial t}\left(\nabla^2 B - \mu_0 \frac{ne^2}{m} B \right) = 0$$

$$\nabla^2 \dot{B} - \frac{1}{\lambda^2}\dot{B} = 0 \qquad \lambda = \sqrt{\frac{m}{\mu_0 ne^2}}$$

$$\dot{B} = \dot{B}_a \exp\left(\frac{-x}{\lambda} \right)$$

The London theory

To match the experimental result

$$\nabla^2 \dot{B} - \frac{1}{\lambda^2}\dot{B} = 0 \quad \Longrightarrow \quad \boxed{\nabla^2 B - \frac{1}{\lambda^2} B = 0}$$

$$\boxed{\nabla \times J + \frac{ne^2}{m} B = 0} \qquad \boxed{\frac{\partial J}{\partial t} = \frac{ne^2}{m} E}$$

These equations are called London equations

To solve $B(x)$ and apply the boundary condition

$$\Longrightarrow \quad B = B_a \exp\left(\frac{-x}{\lambda_L}\right)$$

λ_L is called London penetration depth

Surface current

Any current

$$\nabla \times B = \mu_0 J$$ ⇒

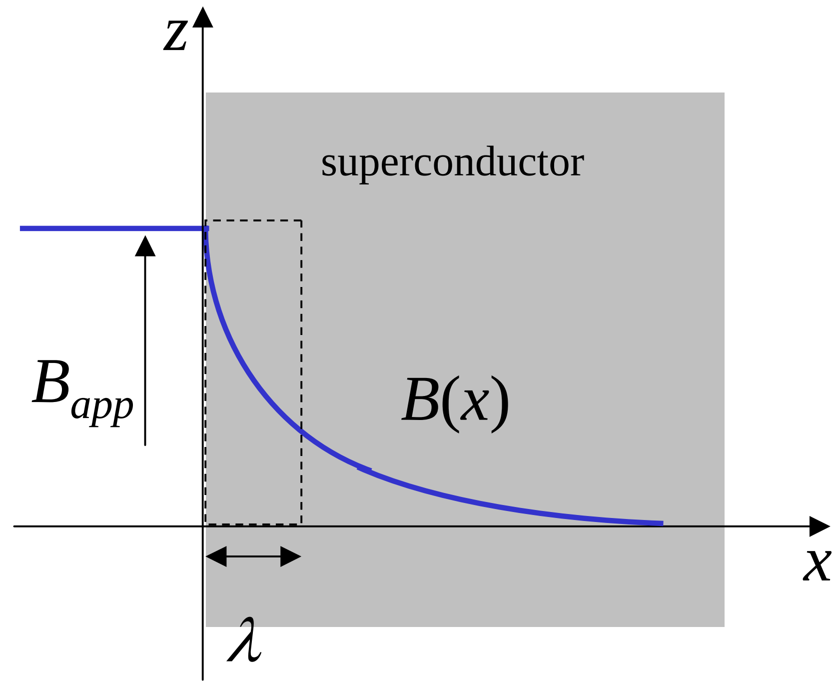
The field has only *z*-component

$$-\frac{\partial B}{\partial x} = \mu_0 J_y$$

from

$$B = B_a \exp\left(\frac{-x}{\lambda_L}\right)$$

$$J_y = \frac{B_a}{\mu_0 \lambda_L} \exp\left(\frac{-x}{\lambda_L}\right)$$



$\lambda_L \sim n^{1/2}$ ⇒

When T approaches T_c
λ_L rises to infinity

Consider the two-fluid model

$$J = J_n + J_s$$

$$J_n = \sigma' E$$

$$\frac{\partial J_s}{\partial t} = \frac{1}{\mu_0 \lambda_L^2} E$$

$$\nabla \times J_s + \frac{1}{\mu_0 \lambda_L^2} B = 0$$

A superconducting slab

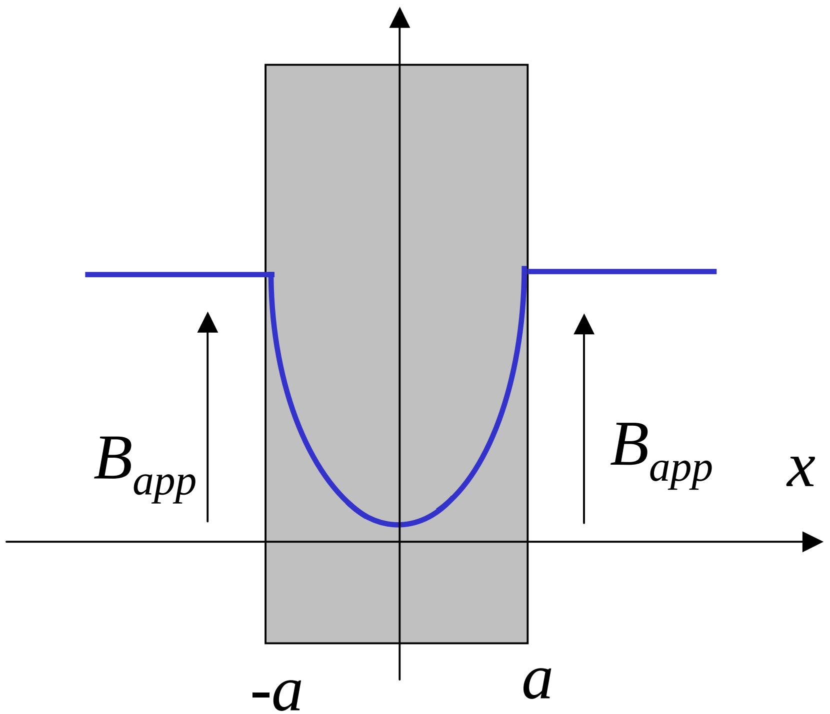
For uniform applied field

$$\frac{\partial B}{\partial y} = \frac{\partial B}{\partial z} = 0$$

$$\frac{\partial^2 B}{\partial x^2} = \frac{1}{\lambda_L^2} B$$

the boundary condition

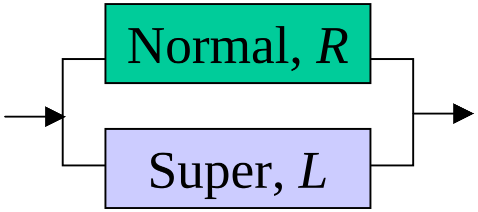
$$B = B_a \qquad x = \pm a$$



$$B = B_a \left(e^{x/\lambda_L} + e^{-x/\lambda_L} \right) = B_a \frac{\cosh\left(x/\lambda_L \right)}{\cosh\left(a/\lambda_L \right)}$$

High-frequency dissipation

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau} = \frac{ne^2\tau}{m} + i\frac{ne^2}{m\omega}$$



For AC current

$$\frac{J_s}{J_n} = \frac{R}{\left|1/i\omega L\right|} = \frac{\dfrac{n_s e^2}{m\omega}}{\dfrac{n_n e^2 \tau_n}{m}} = \frac{n_s}{n_n \tau_n \omega}$$

The cross-over frequency is $\omega_0 = \frac{R}{L} = \frac{n_s}{n_n \tau_n}$

If $n_n \sim n_s$, $\tau \sim 10^{-12}$s, $\omega_0 \sim 10^{11}$Hz

The dissipation $\rho J^2 = \mathrm{Re}\left(\frac{1}{\sigma}\right) J^2 = \frac{\sigma_n}{\sigma_n^2 + \sigma_s^2} J^2 \simeq \frac{\sigma_n}{\sigma_s^2} J^2 \propto \omega^2 \sigma_n$

The absorptivity of a superconducting surface

$$\nabla \times B = \mu_0 J$$

Sheet current density $K = 2H_{inc}$

Power dissipated per unit area

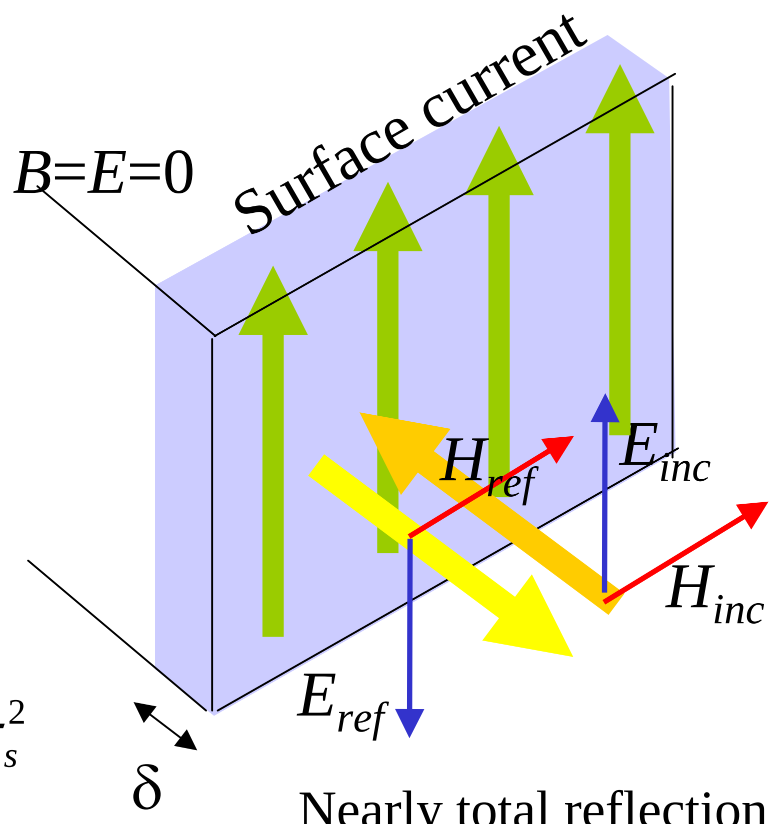
$K^2 R_S$ R_S: surface resistance

The skin depth

$$\delta = \left[\frac{1}{2} \omega \mu_0 \left(|\sigma| + \sigma_s \right) \right]^{-1/2}$$

$$R_S = \delta^{-1} \mathrm{Re}(1/\sigma) = \delta^{-1} \sigma_n / |\sigma|^2 \simeq \delta^{-1} \sigma_n / \sigma_s^2$$

absorptivity $\mathcal{A} = \frac{P_{abs}}{P_{inc}} = \frac{K^2 R_S}{\mu_0 H_{inc}^{\ 2}} = \frac{4R_S}{\mu_0}$



Nearly total reflection

Normal metal v.s. superconductor

Normal metal,

$$\sigma_s = 0 \qquad \delta = \left[\frac{1}{2}\omega\mu_0\sigma_n\right]^{-1/2}$$

$$R_S = \delta^{-1}\,\mathrm{Re}\left(1/\sigma\right) = \delta^{-1}\sigma_n^{-1}$$

$$\mathcal{A} = \frac{4R_S}{\mu_0} = \frac{4}{\sqrt{2}}\sqrt{\frac{\omega}{\mu_0\sigma_n}} \propto \sigma_n^{-1/2}\omega^{1/2}$$

superconductor,

$$\delta = \left[\frac{1}{2}\omega\mu_0\left(|\sigma| + \sigma_s\right)\right]^{-1/2} \simeq \left(\omega\mu_0\sigma_s\right)^{-1/2}$$

$$R_S \simeq \delta^{-1}\,\sigma_n/\sigma_s^2 \simeq \sqrt{\omega\mu_0}\,\sigma_n/\sigma_s^{3/2}$$

$$\mathcal{A} = \frac{4R_S}{\mu_0} = \frac{4\sqrt{\omega\mu_0}\,\sigma_n/\sigma_s^{3/2}}{\mu_0} = 4\sqrt{\frac{\omega}{\mu_0}}\,\frac{\sigma_n}{\sigma_s^{3/2}} \propto \omega^2\sigma_n$$

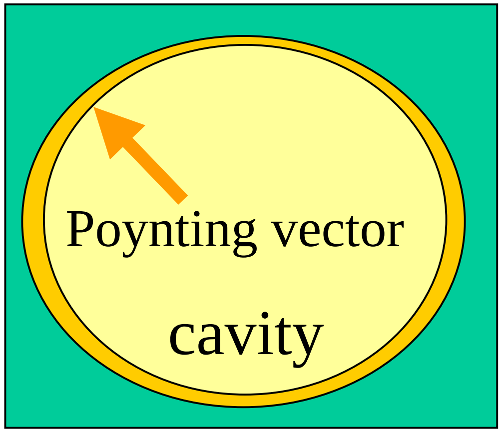
Quality factor

$$Q = \frac{\text{stored energy}}{\text{loss per radian}} = \frac{V\mu_0 H^2/2}{(\mu_0/\omega c)H^2 \mathcal{A} S} = \frac{\omega V}{2cS\mathcal{A}} \approx \frac{L}{\lambda}\frac{1}{\mathcal{A}} \approx \frac{1}{\mathcal{A}}$$

V: cavity volume

S: cavity surface area

L: dimension of the cavity ~λ (resonance)



Poynting vector= $\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$

Incident power per area $\frac{1}{\mu_0}|\mathbf{E}\times\mathbf{B}| = \frac{H^2}{\mu_0 c}$

Normal metal $Q<10^4$ superconductor $Q\sim10^{10}$

High quality microwave filter

Homework

1. Derive $\sigma(\omega) = \dfrac{\sigma_0}{1 - i\omega\tau}$ in which $\sigma_0 = \dfrac{ne^2\tau}{m}$

and show that when $\tau \to \infty$ the impedance becomes “inductive”

2. Estimate the portion of normal current in the two-fluid model when applied an AC electric field at 1MHz. Use the result L(Henry)~$10^{-12}R_N$(Ohm).

3. Estimate the number density of superelectrons by using the London theory. For example, the penetration depth of Pb at zero temperature is experimentally found to be 3.9×10^{-6}cm.