8. The superconducting properties of small specimens

The effect of penetration

$$\mu_0 \frac{H_C^2}{2} = g_n(T,0) - g_s(T,0)$$

➡ The result without the penetration

The flux density decreases to zero with a characteristic distance λ into the interior. In this region, the magnetisation is smaller than $-H$.

H_C is larger than the value given by the equation, especially for a small specimen

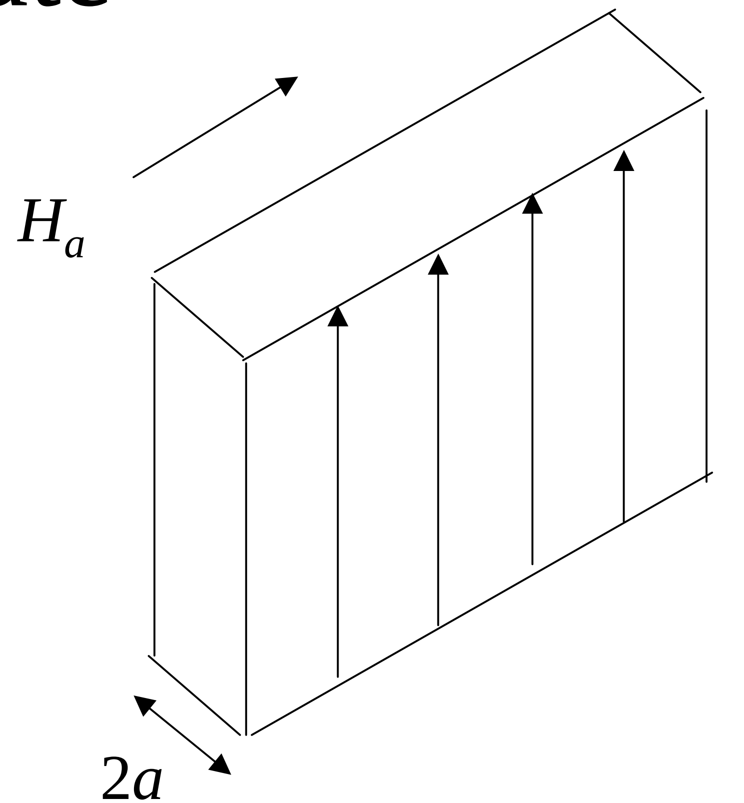
The critical field of a parallel-sides plate

$$B = \mu_0 H_a \frac{\cosh\left(x/\lambda_L\right)}{\cosh\left(a/\lambda_L\right)}$$

$$M = \frac{B}{\mu_0} - H_a = H_a\left[\frac{\cosh\left(x/\lambda_L\right)}{\cosh\left(a/\lambda_L\right)} - 1\right]$$

The total moment per unit area

$$m = \int_{-a}^{a} M dx = -2aH_a\left[1 - \frac{\lambda}{a}\tanh\frac{a}{\lambda}\right] \propto H_a$$



The magnetic contribution to the Gibbs free energy per unit area

$$\Delta G = -\frac{1}{2}\mu_0 m H_a = a\mu_0 H_a^2 \left[1 - \frac{\lambda}{a}\tanh\frac{a}{\lambda}\right]$$

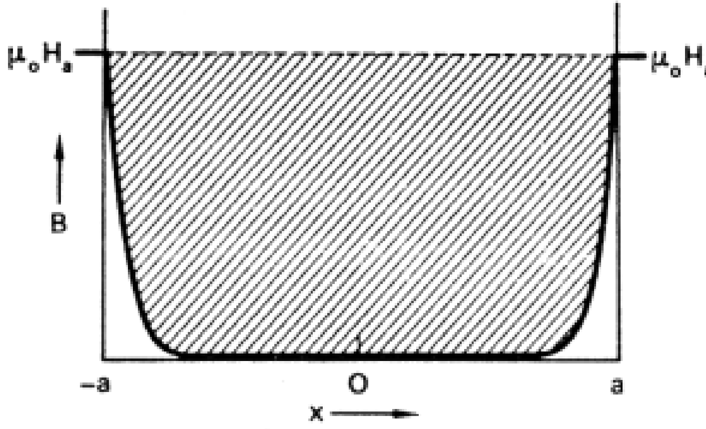
k-factor, Effective susceptibility

$$G_n - G_s = 2a\left(g_n - g_s\right) = a\mu_0 H_C'^2 \left[1 - \frac{\lambda}{a}\tanh\frac{a}{\lambda}\right]$$

$$\mu_0 \frac{H_C^2}{2} = g_n - g_s \quad \Rightarrow \quad H_C^2 = H_C'^2 \left[1 - \frac{\lambda}{a}\tanh\frac{a}{\lambda}\right]$$

$$H_C' = H_C \left[1 - \frac{\lambda}{a}\tanh\frac{a}{\lambda}\right]^{-1/2}$$

$a \gg \lambda$ $$\frac{\lambda}{a}\tanh\frac{a}{\lambda} \sim \frac{\lambda}{a}$$

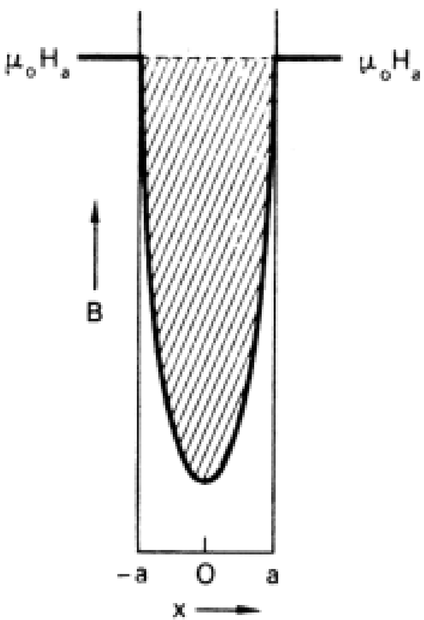


$$H_C' \simeq H_C\left[1-\frac{\lambda}{a}\right]^{-1/2} \simeq H_C\left(1+\frac{2a}{\lambda}\right)$$

$k \sim 1-\frac{\lambda}{a}$ The perfect diamagnetism region shrinks to 2(a-λ)

$a \ll \lambda$ $$\frac{\lambda}{a}\tanh\frac{a}{\lambda} \sim \left(\frac{\lambda}{a}\right)\frac{1}{3}\left(\frac{a}{\lambda}\right)^3 = \frac{1}{3}\left(\frac{a}{\lambda}\right)^2$$

$$H_C' \simeq \sqrt{3}\frac{\lambda}{a}H_C$$



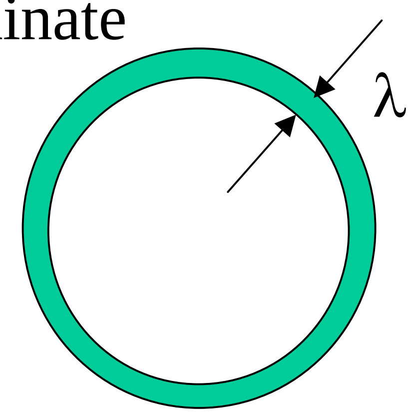
When T approaches T_C, the penetration depth is divergent and this effect is pronounce.

$$\frac{\lambda}{\lambda_0} = \frac{1}{\left(1-t^4\right)}$$

For complicated geometries

For a long cylinder, consider:

$$\nabla^2 B - \frac{1}{\lambda^2} B = 0$$ in cylindrical coordinate



$a \gg \lambda$ $$H'_C \simeq H_C \left(1 + \frac{a}{\lambda}\right)$$

The area of perfect diamagnetism $\pi a^2 - 2\pi a \lambda$

$$\Delta g = -\frac{1}{2}\mu_0 H_a^2$$

The area of penetrated region: $2\pi a \lambda$ $\Delta g = 0$

$$\Delta G = -\frac{1}{2}\mu_0 H_a^2 \left[\pi a^2 - 2\pi a \lambda\right] = -\frac{1}{2}\mu_0 H_a^2 \pi a^2 \boxed{\left(1 - \frac{2\lambda}{a}\right)}$$ *k*-factor

$a \ll \lambda$ $$H'_C \simeq \sqrt{8}\frac{\lambda}{a} H_C$$

Limitation of the London theory

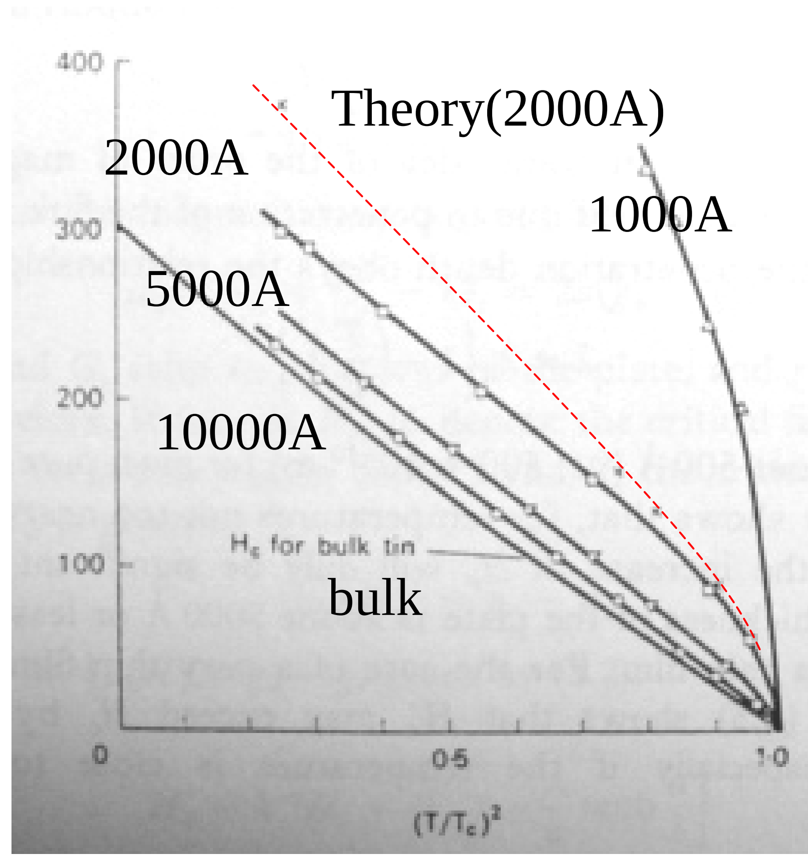
$a \ll \lambda \qquad H'_C \propto \frac{\lambda}{a} H_C$

Reducing a will increase H_C

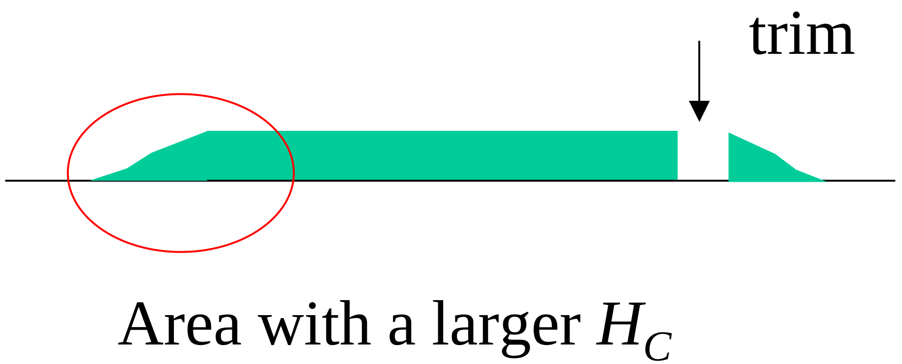
In London's theory, the penetration depth is independent of field and dimension of the specimen.

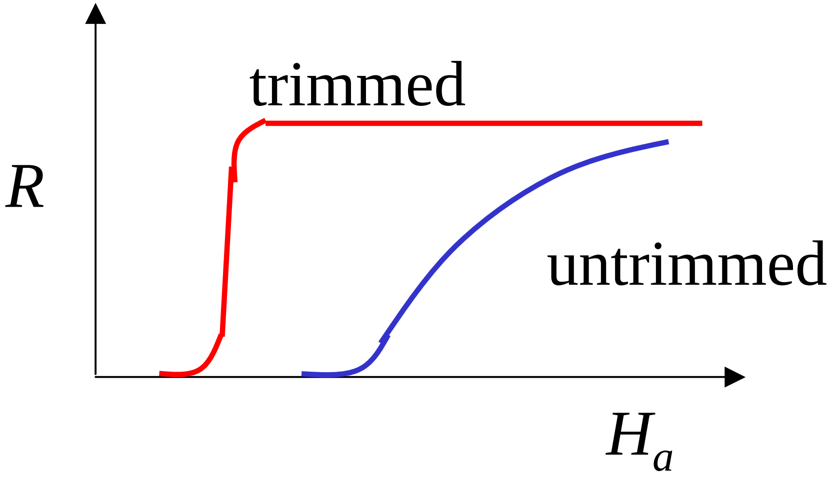
A weak field theory

The importance of the coherent length



Edge effects



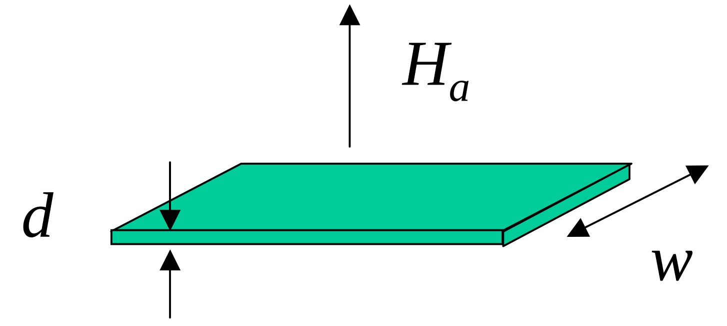


Transitions in perpendicular fields

The effect of demagnetizing factor

$$n \simeq 1 - \frac{d}{w}$$

$$H_C(1-n) \simeq \frac{d}{w} H_C$$



When d is small, H_C is almost zero

In fact, the effect of penetration depth will modify the result → The intermediate state becomes the mixed state

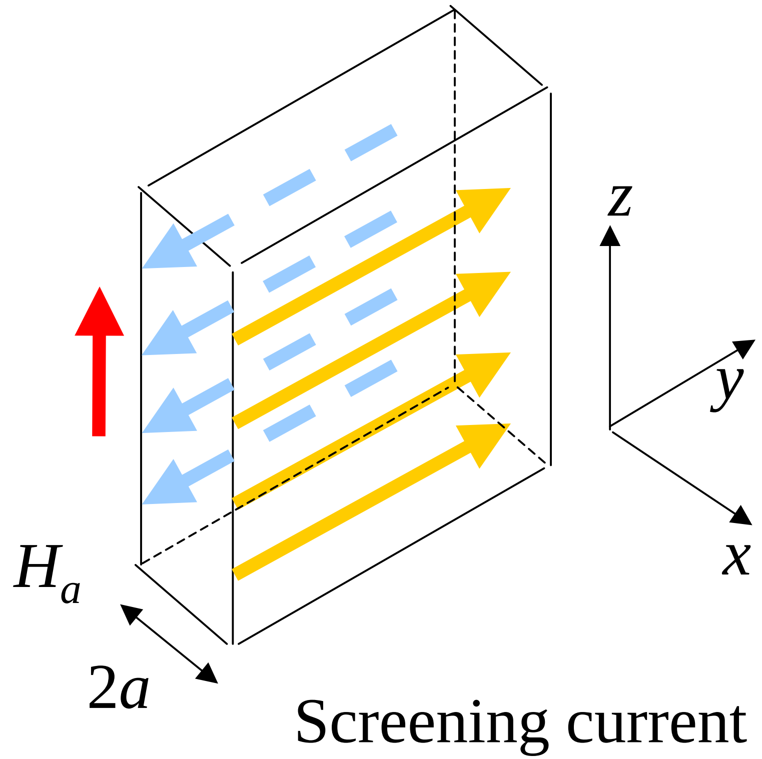
A small component perpendicular to the film will affect the result for a parallel field.

Critical currents of thin specimens

$$B = \mu_0 H_a \frac{\cosh(x/\lambda)}{\cosh(a/\lambda)}$$

$$\nabla \times B = \mu_0 J$$

$$J_y = \frac{1}{\mu_0}\frac{\partial B}{\partial x} = -\frac{H_a}{\lambda}\frac{\sinh(x/\lambda)}{\cosh(a/\lambda)}$$



For thick specimens, $a \gg \lambda$

$$\tanh(a/\lambda) \simeq 1$$

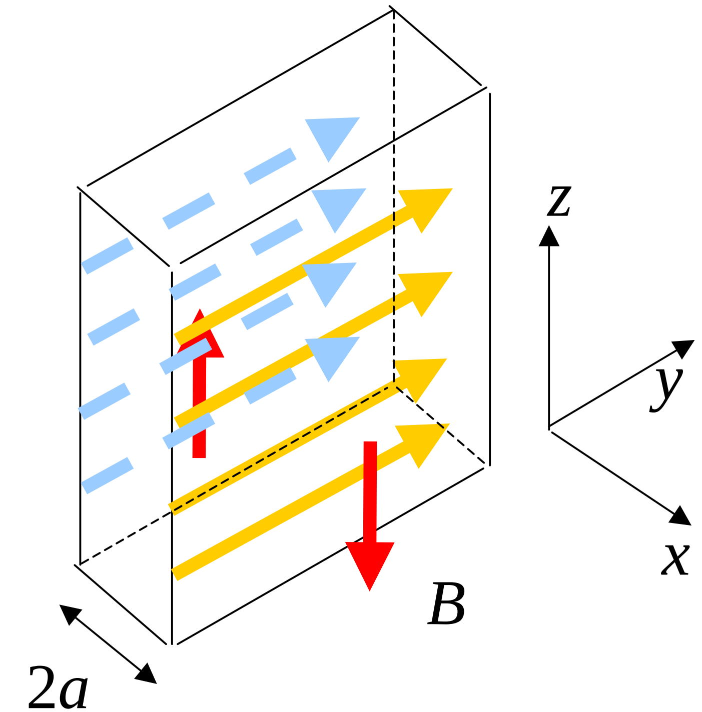
The critical (screening) current $J_C = \frac{H_C}{\lambda}$

Transport current

The symmetry conditions

$$J(x) = J(-x) \qquad B(x) = -B(-x)$$

$$\nabla^2 B - \frac{1}{\lambda^2} B = 0 \qquad \text{London eq.}$$



$$\Rightarrow \quad B \propto \frac{\sinh(x/\lambda)}{\cosh(a/\lambda)}$$

$$\Rightarrow \quad J(x) \propto J(a) \frac{\cosh(x/\lambda)}{\cosh(a/\lambda)}$$

$$B = -\mu_0 \lambda J(a) \frac{\sinh(x/\lambda)}{\cosh(a/\lambda)}$$

Total currents per unit width $\quad K = \int_{-a}^{a} J(x) dx \propto 2\lambda J(a) \tanh(a/\lambda)$

The critical current

$$J_C = \frac{H_C}{\lambda}$$

$$K_C = 2\lambda J_C \tanh\left(a/\lambda\right) = 2H_C \tanh\left(a/\lambda\right)$$

$a \gg \lambda$ $\qquad K_C \simeq 2H_C$

The field strength on the surface is $\quad H = -\lambda J_C = H_C$

Silsbee's rule

$a \sim \lambda$ $\qquad K'_C = 2H_C \tanh\left(a/\lambda\right) = \tanh\left(a/\lambda\right) K_C$

The critical current is reduced!

$a \ll \lambda$ $\qquad K'_C = K_C \tanh\left(a/\lambda\right) \simeq \left(a/\lambda\right) K_C$

The critical current is proportional to the thickness of the slab

The breakdown of Silsbee's rule

The field strength on the surface is

$$H(a) = H_C \tanh(a/\lambda)$$

$$\tanh(a/\lambda) < 1 \quad \Rightarrow$$

$$H_i = H(a) < H_C$$

From previous result $H_C < H_C'$

$$\Rightarrow \quad H_i < H_C < H_C'$$

$a \ll \lambda$

$$H_i = \frac{a}{\lambda} H_C \quad \Rightarrow \quad H_i H_C' = \sqrt{3} H_C^2$$

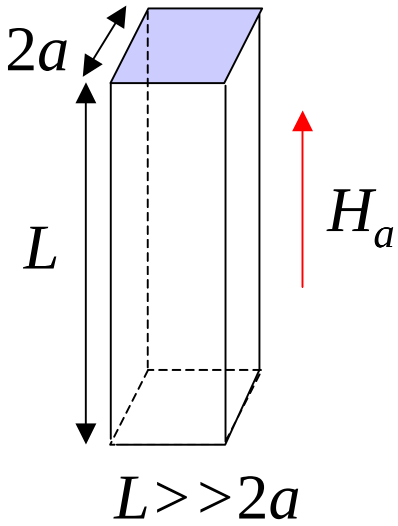
$$H_C' \simeq \sqrt{3} \frac{\lambda}{a} H_C$$

Homework

1. Calculate the correction of critical field, due to the effect of penetration, for a long square bar with a length *a* for each side.

   hint: Solve $\nabla^2 B - \frac{1}{\lambda^2} B = 0$ with the boundary conditions: $B\left(x = \pm a, y\right) = B\left(x, y = \pm a\right) = \mu_0 H_a$

   Then integrate the total moment and calculate the magnetic contribution of the Gibbs free energy



2. Complete the derivation of the flux density $B = -\mu_0 \lambda J(a) \frac{\sinh\left(x/\lambda\right)}{\cosh\left(a/\lambda\right)}$

   for the slab carrying a transport current $J(x)$.