Chapter 9
Ginzburg-Landau theory

The limit of London theory

The London equation $$\nabla \times \mathbf{J} = -\frac{\mathbf{B}}{\mu_0 \lambda^2}$$

The London theory is plausible when

1. The penetration depth is the dominant length scale

   $\lambda \gg l$ mean free path $\lambda \gg \xi_0$ coherent length

2. The field is small and can be treated as a perturbation
3. n_s is nearly constant everywhere

The coherent length should be included in a new theory

Ginzburg-Landau theory

1. A macroscopic theory
2. A phenomenological theory
3. A quantum theory ⟺ London theory is classical

Introduction of pseudo wave function $\Psi(\mathbf{r})$

$\left|\Psi(\mathbf{r})\right|^2$ is the local density of superconducting electrons

$$\left|\Psi(\mathbf{r})\right|^2 = n_s^2(\mathbf{r})$$

The free energy density

The difference of free energy density for normal state and superconducting state can be written as powers of $|\Psi|^2$ and $|\nabla\Psi|^2$

potential energy

Kinetic energy

Ginzburg-Landau free energy density at zero field

$$g_s = g_n + \alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4 + \frac{1}{2m^*}\left|\frac{\hbar}{i}\nabla\Psi\right|^2$$

2nd order phase transition

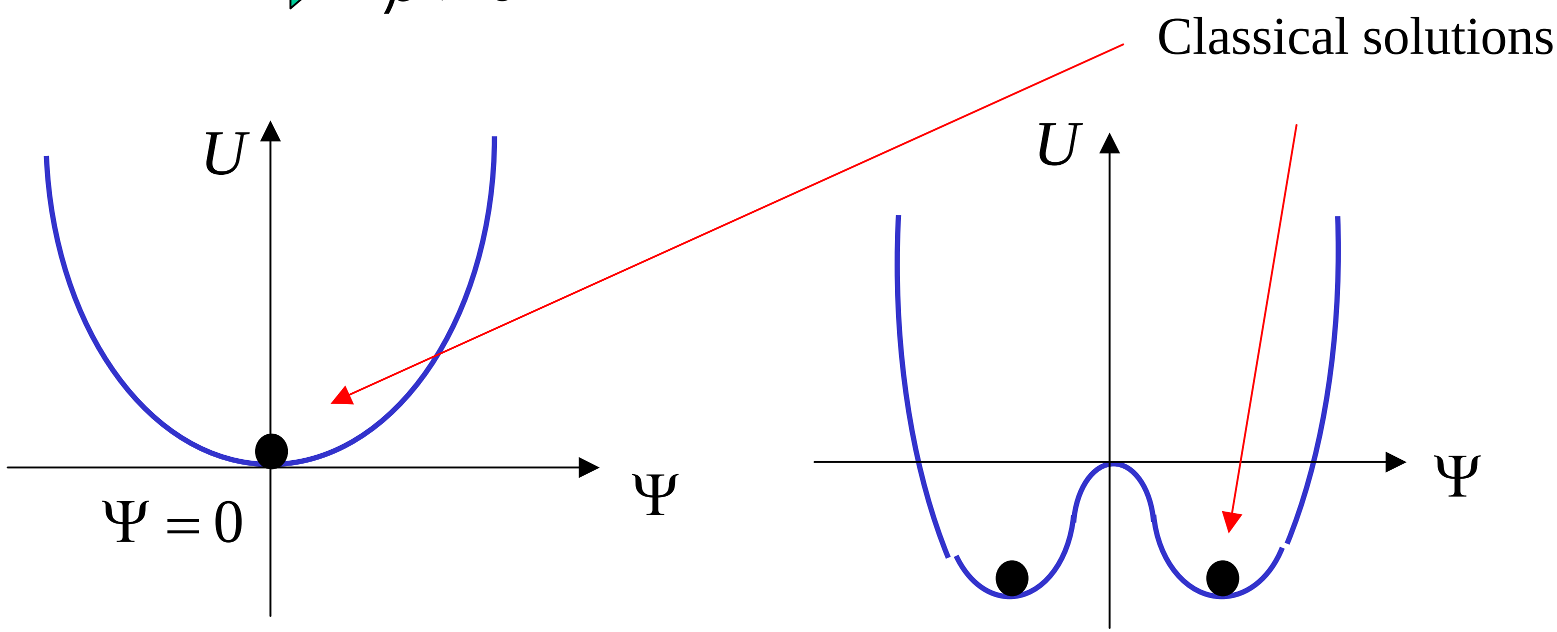
Quantum mechanics

2nd order phase transition

Potential energy $U = \alpha |\Psi|^2 + \frac{\beta}{2} |\Psi|^4$

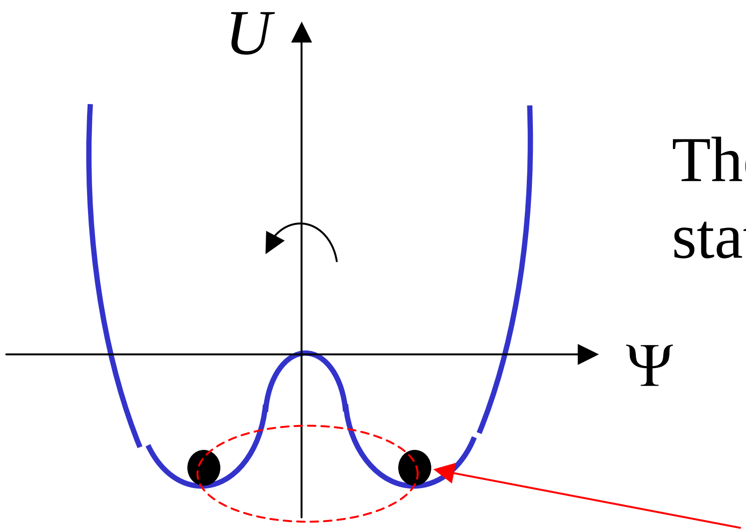
A reasonable theory is bounded, i. e. $U\left(|\Psi| \to \infty\right) \to \infty$

⇒ $\beta > 0$



$\alpha > 0$ Single well $\qquad$ $\alpha < 0$ double well

Spontaneous symmetry breaking



The phase symmetry of the ground state wave function is broken

$$\Psi = |\Psi| e^{i\varphi}$$

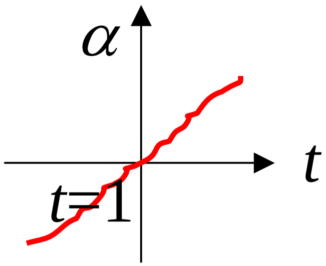
$$|\Psi|^2 = \Psi_\infty^2 = -\frac{\alpha}{\beta}$$

Normal state	Critical point	superconducting state
$\alpha > 0$	$\alpha = 0$	$\alpha < 0$
$\Psi = 0$		$\Psi \neq 0$

$|\Psi|^2$ density of superconducting electrons

The meaning of α

The superconducting critical point is $\alpha = 0$



⟹ $\alpha > 0$ $\alpha = 0$ $\alpha < 0$

$T > T_c$ $T = T_c$ $T < T_c$

Near the critical point, $\alpha = \alpha'(t-1)$

$$t = \frac{T}{T_c}$$

If β is regular near T_c then $|\Psi|^2 = -\frac{\alpha'}{\beta_c}(t-1)$

The London penetration depth is $\lambda_L^2 = \frac{m}{\mu_0 n_s e^2}$

⟹ $\lambda_L \propto \left(\frac{1}{n_s}\right)^{\frac{1}{2}} \propto \frac{1}{(1-t)^{1/2}}$

Consistent with the observation

$$\frac{\lambda_L(T)}{\lambda_L(0)} = \frac{1}{(1-t^4)^{1/2}}$$

Magnetic field contribution

at non zero field, there are two modifications

$$\mathbf{p} \rightarrow \mathbf{p} - e^{*}\mathbf{A}$$

The vector potential

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\Delta g = \frac{1}{2}\mu_0 H^2$$

For perfect diamagnetism

$$\Delta g\left(H_a\right) = -\mu_0 \int_0^{H_a} M dH_a$$

The canonical momentum

The first modification is to include the hamiltonian of a charged particle in a magnetic field

$$\mathbf{E} = -\nabla\phi - \frac{\partial}{\partial t}\mathbf{A}$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

For a charged paticle, $$m\mathbf{v}(t) = m\mathbf{v}(0) + q\int_0^t \mathbf{E}dt$$
$$= m\mathbf{v}(0) - q\mathbf{A}$$

$m\mathbf{v}(t) + q\mathbf{A} = m\mathbf{v}(0)$ is conserved in the magnetic field

The canonical momentum is chosen as $\mathbf{p}_{\text{canonical}} = m\mathbf{v} + q\mathbf{A}$

The kinetic energy is $\frac{1}{2}m\mathbf{v}^2 = \frac{1}{2m}\left(\mathbf{p}_{\text{canonical}} - q\mathbf{A}\right)^2$

Gauge transformation

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla \chi$$

$$\phi \to \phi' = \phi - \frac{\partial}{\partial t}\chi$$

$$\mathbf{E} = -\nabla \phi - \frac{\partial}{\partial t}\mathbf{A}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

The physics is unchanged

The phase of the particle wave function will be changed by a phase factor

$$\Psi(\mathbf{r}) \to \Psi'(\mathbf{r}) = \Psi(\mathbf{r}) \exp\left(\frac{ie}{\hbar}\chi\right)$$

$$\left(\mathbf{p} - e\mathbf{A}'\right)\Psi'(\mathbf{r}) = \left(-i\hbar\nabla - e\mathbf{A}'\right)\left\{\Psi \exp\left(\frac{ie}{\hbar}\chi\right)\right\}$$

$$H = \frac{1}{2m}\left(\mathbf{p} - e\mathbf{A}\right)^2 + U$$

$$= \exp\left(\frac{ie}{\hbar}\chi\right)\left\{\left(-i\hbar\nabla - e\mathbf{A}'\right)\Psi + \left(\nabla\chi\right)\Psi\right\}$$

$$H\Psi = H'\Psi'$$

$$= \exp\left(\frac{ie}{\hbar}\chi\right)\left(-i\hbar\nabla - e\mathbf{A}\right)\Psi$$

Comment: not all theory are gauge-invariant, the theory keeps gauge-invariance is called a gauge theory

The meaning of $|\Psi|^2$

Energy density

$$\frac{1}{2m^*}\left|\left(\frac{\hbar}{i}\nabla - e^* A\right)\Psi\right|^2 = \frac{1}{2m^*}\left|\left(\frac{\hbar}{i}\nabla\left|\Psi\right| + \hbar\left|\Psi\right|\nabla\varphi - e^* A\left|\Psi\right|\right)e^{i\varphi}\right|^2$$

Im. part　　Real part

with $\Psi = \left|\Psi\right|e^{i\varphi}$

$$= \frac{1}{2m^*}\left\{\hbar^2\left(\nabla\left|\Psi\right|\right)^2 + \left(\hbar\nabla\varphi - e^* A\right)^2\left|\Psi\right|^2\right\}$$

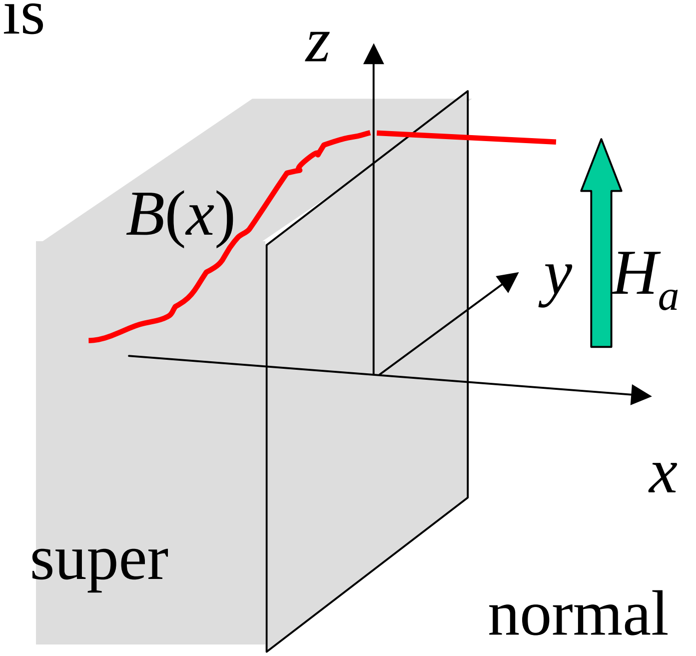
- The first term arises when the number density n_s has a non-zero gradient, for example near the N-S boundary (the length scale is coherent length ξ, and in type I SC, $\xi << \lambda$)
- The second term is the kinetic term associated with the supercurrent. If the phase is constant of position, it gives

$$= \frac{e^{*2}A^2\left|\Psi\right|^2}{2m^*}$$

Penetration near the N-S boundary

Near the surface, the magnetic induction is

$$B_z(x) \sim B_z(0)\, e^{x/\lambda} \quad \text{for } x<0$$



$$\mathbf{B} = \nabla \times \mathbf{A}$$

Choose the gauge

$$A_y(x) = \lambda B_z(x) \qquad B_z(x) = -\frac{\partial A_y}{\partial x}$$

We found the energy density

$$\frac{e^{*2} A^2 |\Psi|^2}{2m^*} = \frac{e^{*2} \lambda^2 B^2 |\Psi|^2}{2m^*}$$

Should be equal to the field energy density $\dfrac{B^2}{2\mu_0}$

$$\lambda^2 = \frac{m^*}{e^{*2} \mu_0 |\Psi|^2}$$

From London's theory $\lambda^2 = \dfrac{m^*}{e^{*2} \mu_0 n_S}$

Kinetic energy density= $n_S\left(\frac{1}{2}m^*v_S^2\right)$

The supercurrent velocity= $m^*\mathbf{v}_S = \mathbf{p}_S - e^*\mathbf{A}$

$$= \hbar\nabla\varphi - e^*\mathbf{A}$$

For $\hbar\nabla\varphi = 0$

$$n_S\left(\frac{1}{2}m^*v_S^2\right) = n_S\frac{\left|e^*\mathbf{A}\right|^2}{2m^*}$$

While in GL theory, the energy density $= \frac{e^{*2}A^2\left|\Psi\right|^2}{2m^*}$

$$n_S = \left|\Psi\right|^2$$

The meaning of the wavefunction Ψ

GL theory and London theory

In bulk superconductors

$$g_s - g_n = -\frac{\mu_0}{2} H_C^2$$

$$= \alpha \left|\Psi_\infty\right|^2 + \frac{\beta}{2}\left|\Psi_\infty\right|^4$$

$$= -\frac{\alpha^2}{2\beta}$$

In previous discussion, we have(in bulk) $n_S = \left|\Psi_\infty\right|^2 = \frac{-\alpha}{\beta}$

with $\lambda^2 = \frac{m^*}{e^{*2}\mu_0 n_S}$ we have

$$\alpha = \frac{-\mu_0 H_C^2}{n_S} = \frac{-\mu_0^2 e^{*2} \lambda^2 H_C^2}{m^*}$$

$$\beta = \frac{\mu_0^3 e^{*4} \lambda^4 H_C^2}{m^{*2}}$$

The temperature dependences near critical point

Near the critical point $\lambda \propto \dfrac{1}{1-t^4}$ $\qquad t = \dfrac{T}{T_C}$

$$\left|\Psi_\infty\right|^2 = n_S = \frac{m^*}{e^{*2}\mu_0\lambda^2} \propto 1-t^4 \simeq 1-t$$

$$\varepsilon \to 0$$

$$1-t^4 = 1-(1-\varepsilon)^4 \simeq 1-(1-4\varepsilon) \simeq 4\varepsilon = 4(1-t)$$

$$H_C \simeq H_C(0)\left(1-t^2\right)$$

$$\alpha \propto \lambda^2 H_C^2 \propto \frac{\left(1-t^2\right)^2}{1-t^4} \simeq \frac{(2\varepsilon)^2}{4\varepsilon} \simeq 1-t$$

$$\beta \propto \lambda^4 H_C^2 \propto \frac{\left(1-t^2\right)^2}{\left(1-t^4\right)^2} \simeq \frac{(2\varepsilon)^2}{(4\varepsilon)^2} = \text{constant of } t$$

⇨ Parameters in GL theory can be determined by $\lambda(T)$ and $H_C(T)$

GL differential eqns

The solution for minimizing g_s in absence of the field, boundary and current is

$$\Psi = \Psi_\infty$$

In general cases, the wavefunction can be written as

$$\Psi = \Psi(\mathbf{r})$$

By variational method

$$\delta \int_V g_S dV = 0$$

We have

$$\alpha\Psi + \beta|\Psi|^2\Psi + \frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - e^*\mathbf{A}\right)^2\Psi = 0 \quad \text{(1st eq)}$$

$$\mathbf{J} = \frac{e^*\hbar}{2m^* i}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) - \frac{e^{*2}}{m^*}\mathbf{A}|\Psi|^2 \quad \text{(2nd eq)}$$

$$= \frac{e^*}{m^*}\left(\hbar\nabla\varphi - e^*\mathbf{A}\right)|\Psi|^2 = e^*|\Psi|^2\mathbf{v}_S$$

Derivation for GL eqns

$$\delta \int_V g_S \, dV = 0$$

g_s is a function of Ψ and $\nabla\Psi \equiv \left(\partial_1\Psi, \partial_2\Psi, \partial_3\Psi\right)$

With boundary conditions, i.e $\Psi\big|_\Omega = 0$ or $\nabla\Psi\big|_\Omega = 0$

$$\frac{\partial g_S}{\partial \Psi} - \sum_i \partial_i \frac{\partial g_S}{\partial\left(\partial_i \Psi\right)} = 0$$

Euler-Lagrange eq.

$$\Rightarrow \quad \frac{\partial g_S}{\partial \Psi^*} - \sum_i \partial_i \frac{\partial g_S}{\partial\left(\partial_i \Psi^*\right)} = 0$$

$$g_s = g_n + \alpha\left|\Psi\right|^2 + \frac{\beta}{2}\left|\Psi\right|^4 + \frac{1}{2m^*}\left|\left(\frac{\hbar}{i}\nabla - e^*\mathbf{A}\right)\Psi\right|^2 + \frac{\mathbf{B}^2}{2\mu_0} - \mu_0 \mathbf{M}\cdot\mathbf{H}$$

$$\frac{\partial g_S}{\partial \Psi^*} = \alpha\Psi + \beta\left|\Psi\right|^2\Psi + \frac{-e^*\mathbf{A}}{2m^*}\cdot\left(\frac{\hbar}{i}\nabla - e^*\mathbf{A}\right)\Psi$$

$$\sum_i \partial_i \frac{\partial g_S}{\partial\left(\partial_i \Psi^*\right)} = \frac{-1}{2m^*}\left(\frac{\hbar}{i}\right)\sum_i \partial_i \left(\frac{\hbar}{i}\partial_i - e^* A_i\right)\Psi$$

$$= \frac{-1}{2m^*}\left\{\left(\frac{\hbar}{i}\right)^2 \nabla^2\Psi - \frac{\hbar}{i} e^* \mathbf{A}\cdot\nabla\Psi\right\}$$

$$0 = \alpha\Psi + \beta\left|\Psi\right|^2 \Psi + \frac{-e^*\mathbf{A}}{2m^*}\cdot\left(\frac{\hbar}{i}\nabla - e^*\mathbf{A}\right)\Psi + \frac{1}{2m^*}\left\{\left(\frac{\hbar}{i}\right)^2 \nabla^2\Psi - \frac{\hbar}{i} e^* \mathbf{A}\cdot\nabla\Psi\right\}$$

$$= \alpha\Psi + \beta\left|\Psi\right|^2 \Psi + \frac{1}{2m^*}\left\{\left(\frac{\hbar}{i}\right)^2 \nabla^2\Psi - 2\frac{\hbar}{i} e^* \mathbf{A}\cdot\nabla\Psi + e^{*2}\mathbf{A}^2\Psi\right\}$$

$$= \alpha\Psi + \beta\left|\Psi\right|^2 \Psi + \frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - e^*\mathbf{A}\right)^2 \Psi$$

First GL equation

$$\delta \int_V g_S \, dV = 0$$

g_s is a function of A and $\nabla \times \mathbf{A} = \mathbf{B}$

With boundary conditions, i.e $\mathbf{A}\big|_\Omega = 0$ or $\partial_j A_i\big|_\Omega = 0$

$$\Rightarrow \quad \frac{\partial g_S}{\partial A_j} - \sum_i \partial_i \frac{\partial g_S}{\partial\left(\partial_i A_j\right)} = 0$$

Euler-Lagrange eq.

$$g_s = g_n + \alpha \left|\Psi\right|^2 + \frac{\beta}{2}\left|\Psi\right|^4 + \frac{1}{2m^*}\left|\left(\frac{\hbar}{i}\nabla - e^*\mathbf{A}\right)\Psi\right|^2 + \frac{\mathbf{B}^2}{2\mu_0} - \mu_0 \mathbf{M}\cdot\mathbf{H}$$

$$\frac{\partial g_S}{\partial A_j} = \frac{-e^*}{2m^*}\left\{\Psi^*\left(\frac{\hbar}{i}\nabla - e^*\mathbf{A}\right)_j \Psi + \Psi\left(-\frac{\hbar}{i}\nabla - e^*\mathbf{A}\right)_j \Psi^*\right\}$$

$$= \frac{-e^*}{2m^*}\left(\frac{\hbar}{i}\right)\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right)_j + \frac{e^{*2}A_j}{m^*}\left|\Psi\right|^2$$

$$\left(\nabla\times\mathbf{A}\right)^2 = \sum_{lmnqr} \varepsilon_{lmn}\varepsilon_{lqr}\left(\partial_m A_n\right)\left(\partial_q A_r\right)$$

$$\sum_i \partial_i \frac{\partial g_S}{\partial\left(\partial_i A_j\right)} = \frac{1}{2\mu_0}\sum_i\sum_{lmnqr}\varepsilon_{lmn}\varepsilon_{lqr}\partial_i\frac{\partial\left(\partial_m A_n\right)\left(\partial_q A_r\right)}{\partial\left(\partial_i A_j\right)}$$

$$= \frac{1}{2\mu_0}\sum_i\sum_{lmnqr}\varepsilon_{lmn}\varepsilon_{lqr}\partial_i\left\{\delta_{mi}\delta_{nj}\left(\partial_q A_r\right)+\delta_{qi}\delta_{rj}\left(\partial_m A_n\right)\right\}$$

$$= \frac{1}{\mu_0}\sum_i\sum_{lmn}\varepsilon_{lmn}\varepsilon_{lij}\partial_i\left(\partial_m A_n\right) = \frac{1}{\mu_0}\sum_i\varepsilon_{ijl}\partial_i\left(\nabla\times\mathbf{A}\right)_l$$

$$= \frac{-1}{\mu_0}\left(\nabla\times\nabla\times\mathbf{A}\right)_j$$

$$\frac{1}{\mu_0}\nabla\times\nabla\times\mathbf{A} = \frac{1}{\mu_0}\nabla\times\mathbf{B} = \mathbf{J}$$

$$0 = \frac{-e^*}{2m^*}\left(\frac{\hbar}{i}\right)\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) + \frac{e^{*2}\mathbf{A}}{m^*}\left|\Psi\right|^2 + \frac{1}{\mu_0}\nabla\times\nabla\times\mathbf{A}$$

$$\mathbf{J} = \frac{e^*\hbar}{2m^* i}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) - \frac{e^{*2}\mathbf{A}}{m^*}\left|\Psi\right|^2$$ second GL equation

Boundary conditions

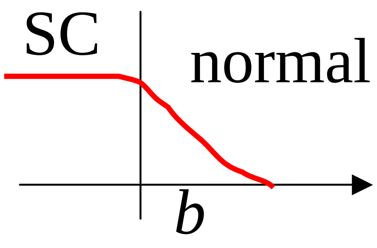
The GL eqns are derived by assuming boundary conditions that

$$\left(\frac{\hbar}{i}\nabla - e^{*}\mathbf{A}\right)\Psi\Big|_{\Omega} = 0 \qquad \text{and} \qquad \mathbf{J}\big|_{\Omega} = 0$$

These are true for a SC-insulator boundary, but not correct for an N-SC boundary

The boundary condition for N-SC is derived by de Gennes using a microscopic theory:

$$\left(\frac{\hbar}{i}\nabla - e^{*}\mathbf{A}\right)\Psi\Big|_{\Omega} = \frac{i\hbar}{b}\Psi$$



Thus the wavefunction will “leak” into the normal region with a characteristic length, *b*. This is called **proximity effect**.

GL coherent length

At zero field, H=0

$$\mathbf{J}=0 \qquad \Psi^*\nabla\Psi-\Psi\nabla\Psi^*=0 \qquad \text{and}$$

$$\nabla\varphi=0$$ Superconducting phase is constant of position

(GL eq 1)

$$\alpha\Psi+\beta|\Psi|^2\Psi-\frac{\hbar^2}{2m^*}\nabla^2\Psi=0$$

$$f\equiv\frac{\Psi}{\Psi_\infty}$$

In 1D system

$$\Psi_\infty\equiv-\frac{\alpha}{\beta}$$

$$-\frac{\hbar^2}{2m^*}\frac{d^2}{dx^2}f+\alpha f+\beta|\Psi_\infty|^2 f^3=0$$

$$\boxed{-\frac{\hbar^2}{2m^*|\alpha|}}\frac{d^2}{dx^2}f+f-f^3=0$$

Dimension=[L^2]

A length scale can be defined $$\frac{\hbar^2}{2m^*|\alpha|} = \xi^2$$

$$-\xi^2 \frac{d^2}{dx^2} f + f - f^3 = 0$$

since $\alpha \propto 1-t$ $$\xi^2 \propto \frac{1}{1-t}$$

Consider the situation that *f*~1 (deep in the SC)

We can expand the GL eq: $f = 1-g \quad g \simeq 0$

$$-\xi^2 \frac{d^2}{dx^2} g + (1+g) - (1+g)^3 = 0$$

$$-\xi^2 g'' - 2g = 0 \qquad g(x) \simeq e^{\pm\sqrt{2}x/\xi}$$

Exact solution

$$-\xi^2 \frac{d^2}{dx^2} f + f\left(1 - f^2\right) = 0$$

The solution $\quad f = \dfrac{e^u - e^{-u}}{e^u + e^{-u}} = \tanh u \qquad u = \dfrac{x}{\sqrt{2}\xi}$

$$\left(\text{with} \quad \frac{df}{du} = \frac{1}{\cosh^2 u} \qquad \frac{d^2 f}{du^2} = -2\frac{\tanh u}{\cosh^2 u} \right)$$

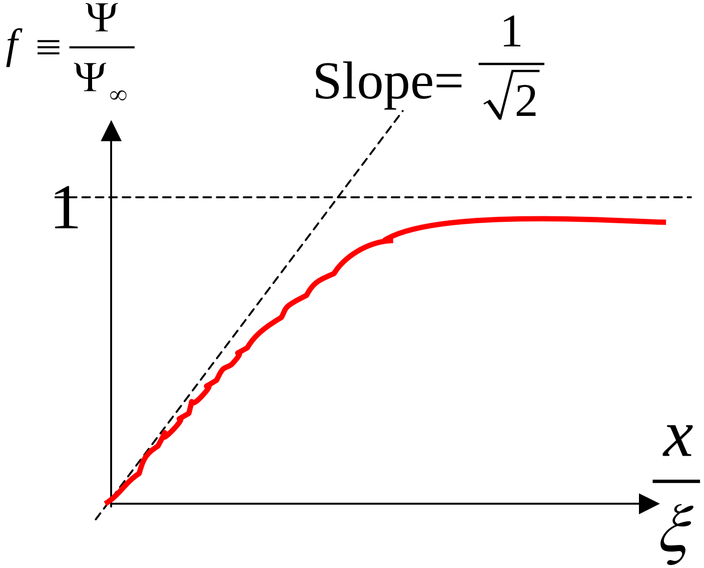
When u is large

$$f = \tanh u = \frac{e^u\left(1 - e^{-2u}\right)}{e^u\left(1 + e^{-2u}\right)}$$

$$= \left(1 - e^{-2u}\right)\left(1 - e^{-2u} + \cdots\right)$$

$$\simeq 1 - 2e^{-2u} = 1 - 2e^{-\sqrt{2}x/\xi}$$

When u is small $\quad f = \tanh u \simeq u = \dfrac{x}{\sqrt{2}\xi}$



Dimensionless GL parameter

$$\alpha = \frac{-\mu_0^2 e^{*2} \lambda^2 H_C^2}{m^*}$$

$$\frac{\hbar^2}{2m^* |\alpha|} = \xi^2$$

$$\xi = \frac{\hbar}{\sqrt{2}\mu_0 e^* \lambda H_C} = \frac{\Phi_0}{2\sqrt{2}\pi\mu_0 H_C \lambda}$$

$$\Phi_0 = \frac{h}{e^*}$$ The fluxoid

$$\kappa = \frac{\lambda}{\xi} = \frac{2\sqrt{2}\pi\mu_0 H_C \lambda^2}{\Phi_0}$$

$$\propto \frac{1-t^2}{1-t^4} = \frac{1}{1+t^2}$$

When $\kappa > \frac{1}{\sqrt{2}}$ type 2 SC

$\kappa < \frac{1}{\sqrt{2}}$ type 1 SC

London penetration depth

(GL eq 2) $$\mathbf{J} = \frac{e^*}{m^*}\left(\hbar\nabla\varphi - e^*\mathbf{A}\right)\left|\Psi\right|^2$$

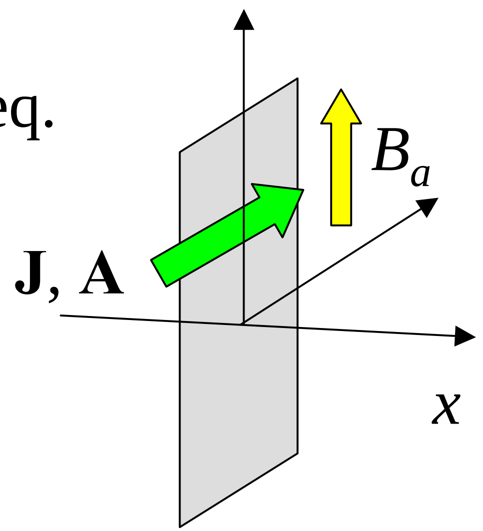
If $\nabla\varphi = 0$ $\left(\text{for } x \gg \xi\right)$ $$\mathbf{J} = -\frac{e^{*2}\mathbf{A}}{m^*}\left|\Psi\right|^2$$

$$\nabla\times\mathbf{J} = -\frac{e^{*2}}{m^*}\left|\Psi\right|^2 \nabla\times\mathbf{A} = -\frac{e^{*2}}{m^*}\left|\Psi\right|^2 \mathbf{B} = -\frac{1}{\mu_0\lambda^2}\mathbf{B}$$

From Ampere's law $\nabla\times\mathbf{B} = \mu_0\mathbf{J}$ $\nabla\cdot\mathbf{B} = 0$

$$\mu_0\nabla\times\mathbf{J} = \nabla\times\nabla\times\mathbf{B} = \nabla\left(\nabla\cdot\mathbf{B}\right) - \nabla^2\mathbf{B} = -\nabla^2\mathbf{B}$$

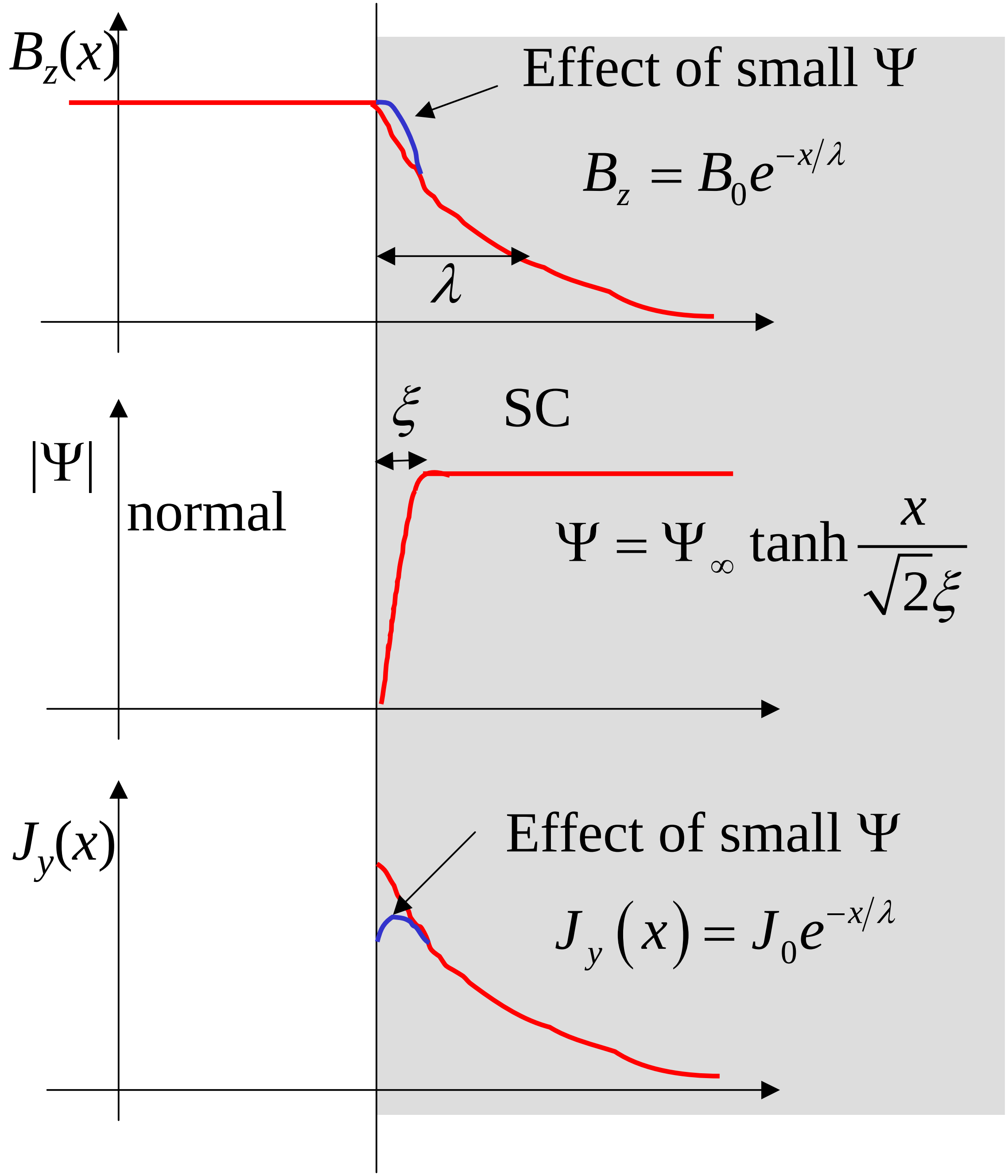
$$\nabla^2\mathbf{B} = \frac{1}{\lambda^2}\mathbf{B}$$ We get the London eq.



$$B_z = B_0 e^{-x/\lambda} = -\frac{A_0}{\lambda}e^{-x/\lambda}$$

$$J_y\left(x\right) = \frac{A_0}{\mu_0\lambda^2}e^{-x/\lambda}$$ $x \gg \xi$

Two length scales, ξ and λ



Type 2 SC

$0 \ll \xi \ll \lambda$