Chapter7　　功和機械能

§7-1 定力所作的功 W

△見圖 7-1 所示，定力所做的功 W 定義為在位移方向的力分量與位移的乘積，即$W \equiv (F\cos\theta)S = \vec{F}\cdot\vec{S}$，其中$\theta$表力方向與位移方向之夾角。因為$W = FS\cos\theta$　$\therefore \cos\theta > 0$時，W 為正，反之為負。

FIGURE 7-1 A particle is moved from A to B by a constant force **F** acting at an angle θ to the displacement **d**.

△見圖 7-2(a)所示，當力與位移平行，則$\theta = 0^\circ \Rightarrow W = Fd$。

△見圖 7-2(b)所示，當力與位移垂直，則$\theta = 90^\circ \Rightarrow W = 0$。

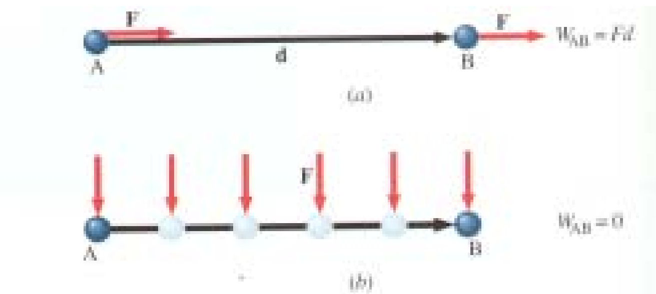


FIGURE 7-2 (a) The work done by a constant force in the same direction as the particle's displacement is *Fd*. (b) The work done by a force perpendicular to the displacement is 0.

△由功之定義知，功之 SI 單位為$N\cdot m = J$。

△例題 7-1。

EXAMPLE 7-1　WORK DONE ON A TOY CHEST

A child is pushing a toy chest across the floor with a constant force of 140 N directed at 30° below the horizontal, as shown in Fig. 7-3*a*.

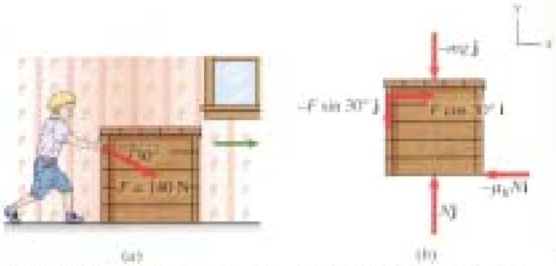


FIGURE 7-3 (*a*) A child pushes a toy chest across the floor with a constant force. (*b*) The free-body diagram of the chest.

The total mass of the chest and its contents is 8.0 kg and the coefficient of kinetic friction between the floor and the chest is 0.75. Calculate the work done by (*a*) the force of the child and (*b*) the frictional force when the chest is pushed a distance of 2.0 m.

§7-2 變力所作的功

△見圖 7-4 所示。

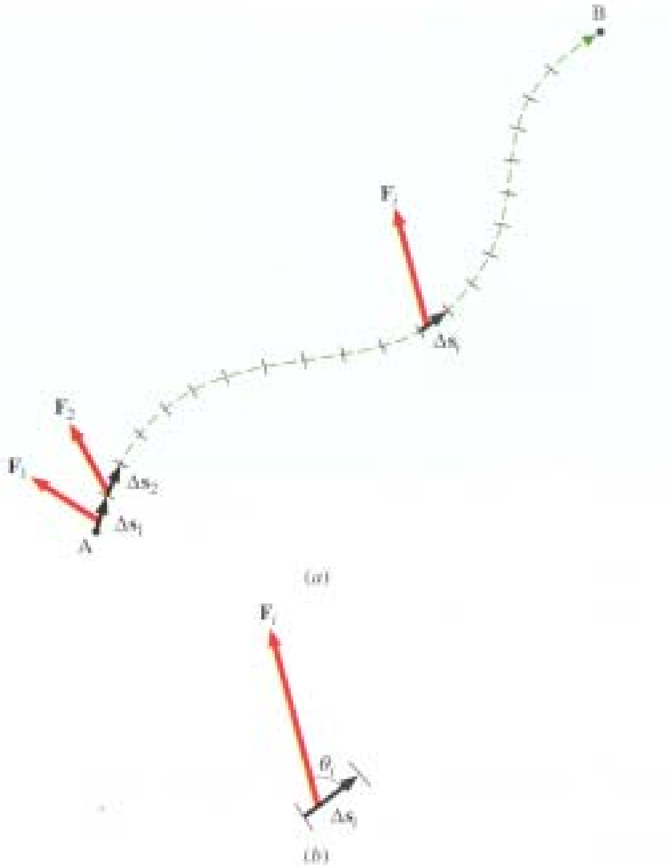


FIGURE 7-4 (*a*) The work done by force **F** is calculated by dividing the path into very small segments and summing the work done over the segments. (*b*) The *i*th segment enlarged.

由定力所作的功之定義知

$\Delta W_i \approx \vec{F}_i \cdot \Delta\vec{S}_i = F_i \Delta S_i \cos\theta_i \;\therefore W_{AB} = \lim_{\Delta s_i \to 0} \sum_i \vec{F}_i \cdot \Delta\vec{S}_i = \lim_{\Delta s_i \to 0} \sum_i F_i \Delta S_i \cos\theta_i$，若假

設一質點受一力 F(x)之作用，而沿 X 軸方向運動，則 $\vec{F}_i = F(x_i)\vec{i}$,

$\Delta\vec{S}_i = \Delta X_i \vec{i}$ ， $\cos\theta_i = 1$， 見圖 7-5 所示。

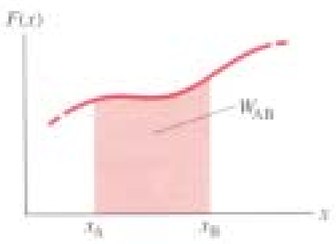


FIGURE 7-5 The work of $F(x)$ between x_A and x_B is the area under the curve representing $F(x)$ between x_A and x_B.

則 $W_{AB} = \lim_{\Delta x_i \to 0} \sum_i F(x_i)\Delta X_i = \int_A^B F(x)dx$， 因此可知在 $F(x)$ 對 x 函數關係圖中，其面積乃代表力 $F(x)$ 所作的功，見圖 7-5。同理對於三度空間而言，力 $\vec{F}$ 所作的功

$$W_{AB} = \int_{r_i}^{r_f} \vec{F}\cdot d\vec{r} = \int_{r_i}^{r_f} (F_x\vec{i} + F_y\vec{j} + F_z\vec{k})\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k}) = \int_{x_i}^{x_f} F_x dx + \int_{y_i}^{y_f} Fydy + \int_{z_i}^{z_f} Fzdz$$

§常見的力所作的功

△彈力 $F = -KX$ 所作的功:見圖 7-7 所示,

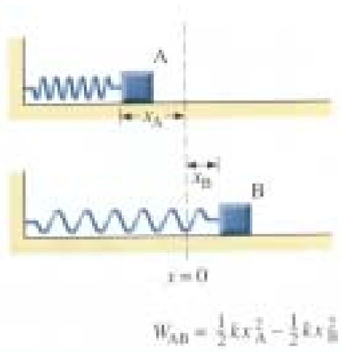


FIGURE 7-7 An object attached to a spring moves from A to B. The work of the spring force only depends on the initial and final positions.

$$W_{AB} = \int_A^B \vec{F}\cdot d\vec{X} = \int_A^B -KXdX = -\frac{1}{2}KX^2\Big|_A^B = \frac{1}{2}KX_A^2 - \frac{1}{2}KX_B^2$$

△重力 $m\vec{g}$ 所作的功；

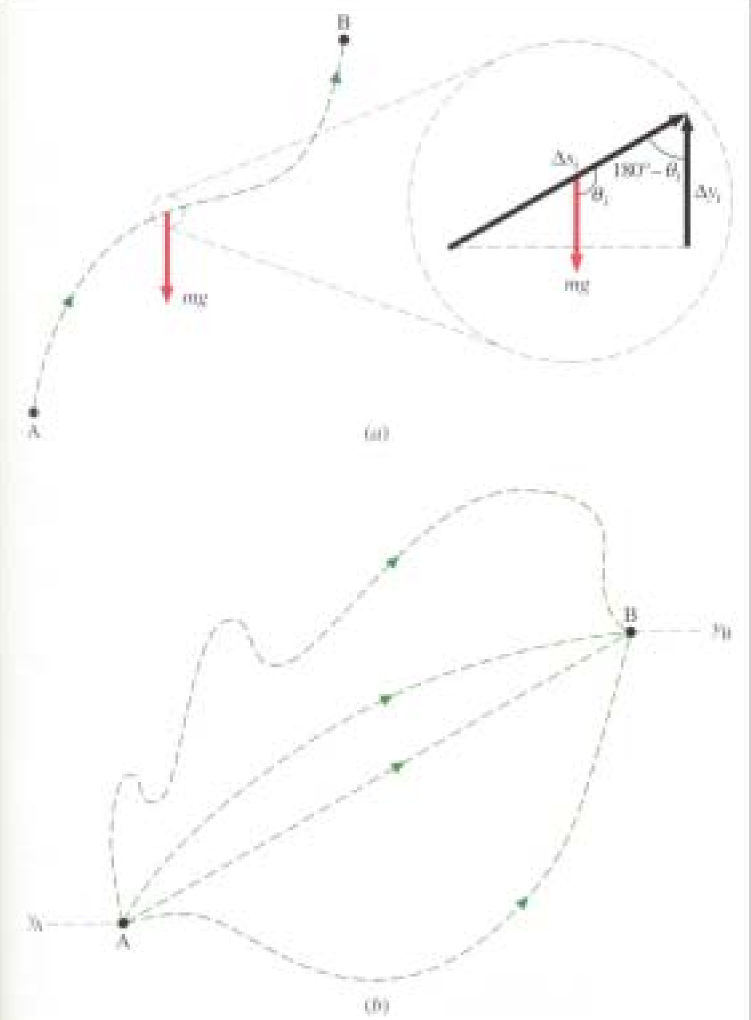


FIGURE 7-8 (*a*) A particle moves from A to B while subjected to the gravitational force *mg*. (*b*) Along any of the paths shown, the work done by *mg* is $-mg(y_B - y_A)$.

見圖 7-8 所示，表一重力(定力)$m\vec{g}$ 對一質量 m 之質點作用，且從 A 沿著任一路徑至 B 點。

$$\therefore W_{AB} = \lim_{\Delta S_i \to 0} \sum_i mg\cos\theta \Delta S_i = \lim_{\Delta S_i \to 0} \sum_i mg[-\Delta S_i \cos(180^\circ - \theta_i)]$$

$$= -mg \lim_{\Delta S_i \to 0} \sum_i \Delta S_i \cos(180^\circ - \theta_i) \text{ (見圖 7-8(a)之圓)}$$

$$= -mg \lim_{y_i \to 0} \sum_i \Delta y_i = -mg(\lim_{\Delta y_i \to 0} \sum_i \Delta y_i = -mg(y_B - y_A))$$

或利用積分法求 $W_{AB} = \int_A^B mg(-\vec{j}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$

$$= \int_A^B -mgdy = -mg\int_A^B dy = -mg(y_B - y_A)$$

在以上推導中，我們假設 A→B 路徑其 g 為定值

△萬有引力 $\vec{F} = -\dfrac{GMm}{r^2}\hat{r}$ 所作的功:

設有一質點 m，在 M 所建立的重力場中，由 A 點移動至 B 點，見

圖 7-9 所示，

FIGURE 7-9 A particle of mass m moves from A to B while subjected to the gravitational force $-(GMm/r^2)\hat{r}$ of the mass M at the origin. The work done depends on the values of r_A and r_B and not on the path.

$$W_{AB} = \lim_{\Delta S_i \to 0} \sum_i \vec{F}_i \cdot \Delta \vec{S}_i = \lim_{\Delta S_i \to 0} \sum_i -\frac{GMm}{r_i^2}\hat{r}_i \cdot \Delta \vec{S}_i$$

$$= \lim_{\Delta r_i \to 0} \sum_i -\frac{GMm}{r_i^2}\Delta r_i = -\int_{r_A}^{r_B} \frac{GMm}{r^2} dr$$

$$= \frac{GMm}{r_B} - \frac{GMm}{r_A}$$

設 $r_A = R_E$ (地表面)，　$r_B = R_E + h$

則 $W_{AB} = \dfrac{GMm}{R_E + h} - \dfrac{GMm}{R_E} = \dfrac{GMm}{R_E}[\dfrac{1}{1+(h/R_E)} - 1]$

$\because \dfrac{1}{1+\in} = 1 - \in + \in^2 - \in^3 + \ldots\ldots \therefore$ 當 $\in \langle\langle 1$ 時

$\dfrac{1}{1+\in} \approx 1 - \in \therefore \dfrac{1}{1+(\mathrm{h}/\mathrm{R_E})} \approx 1 - \dfrac{h}{R_E}$

$\therefore W_{AB} \approx \dfrac{GMm}{R_E}(1 - \dfrac{h}{R_E} - 1) = -\dfrac{GMmh}{R_E^2} = -mgh$

△例 7-2

EXAMPLE 7-2 **WORK DONE ON A BLOCK MOVING ON A RAMP**

A 40-kg block is attached to a spring whose other end is fixed at the top of a ramp. (See Fig. 7-10a.) Initially, the block is held at point A, where the spring is at its unstretched length. The block is then released and begins to slide down the ramp. Identify the forces on the block and calculate the work done by each force when the block moves to point B, which is a distance 0.10 m down the ramp. Take the spring constant to be 200 N/m, and assume that the force of kinetic friction between the block and the ramp is 40 N.

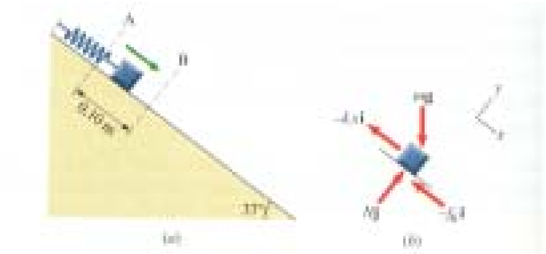


FIGURE 7-10 (a) A block slides from A to B. (b) The forces on the block.

§7-4　功-能定理(Work-energy theorem)

△若作用在某一物體的力很複雜時;則想利用牛頓第二定律來求得,

該物體在某一位置的速度是相當困難的。因此我們可以改採用功-

能定理來解決這一類的問題。

△什麼是功-定理；

即外力和對一質點所作的功等於該質點之動能變化量。

△功-能定理之證明；

[Pf]； $W_{AB} = \int_A^B \vec{F} \cdot d\vec{r} = \int_A^B m\frac{d\vec{v}}{dt} \cdot d\vec{r} = \int_A^B m\frac{d\vec{v}}{dt} \cdot \frac{d\vec{r}}{dt} dt$

$$= m\int_A^B \frac{d\vec{v}}{dt} \cdot \vec{v} dt = m\int_A^B \vec{v} \cdot d\vec{v}$$

$$= m\int_A^B (V_x\vec{i} + V_y\vec{j} + V_z\vec{k}) \cdot (dV_X\vec{i} + dV_y\vec{j} + dV_z\vec{k})$$

$$= m\int_A^B V_x dx + \int V_y dy + \int V_Z dz$$

$$= m[\frac{1}{2}V_x^2 + \frac{1}{2}V_y^2 + \frac{1}{2}V_z^2]|_A^B$$

$$= \frac{1}{2}m(V_X^2 + V_y^2 + V_z^2)|_A^B$$

$$= \frac{1}{2}mV^2|_A^B = \frac{1}{2}mV_B^2 - \frac{1}{2}mV_A^2$$

設動能 $T \equiv \frac{1}{2}mV^2$

$\Rightarrow W_{AB} = T_B - T_A = \Delta T$

△例题 7-4,7-5,7-6,7-7

EXAMPLE 7-4 A Sled on a Hill

Figure 7-12 shows a sled and occupant of total mass 150 kg, which starts from rest at the top of a hill and slides a distance $l = 50.0$ m to the bottom. The hill can be treated as an inclined plane with a constant slope of $\theta = 10.0°$ to the horizontal. The coefficient of kinetic friction between the sled runners and the snow is 0.070. What is the sled's speed at the bottom of the hill?

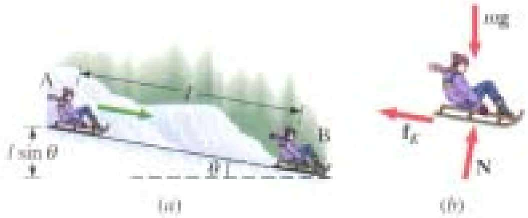


FIGURE 7-12 (*a*) Sled and occupant. (*b*) The forces on the system of the sled and its occupant.

EXAMPLE 7-5 A Particle Subjected to a Variable Force

A particle of mass $m = 0.20$ kg moves in the x direction under the influence of a single force $F(x) = c/x^2$, where $c = 3.0\ \text{N}\cdot\text{m}^2$. The particle's speed at $x_A = 3.0$ m is 7.0 m/s. What is its speed at $x_B = 6.0$ m?

EXAMPLE 7-6 Motion of a Ball at the End of a String

A small ball is hung from the ceiling by a light string of length 1.00 m. (See Fig. 7-13.) If the ball is held at an angle of 30.0° to the vertical and then released, what is its speed when it reaches the lowest point of its arc?

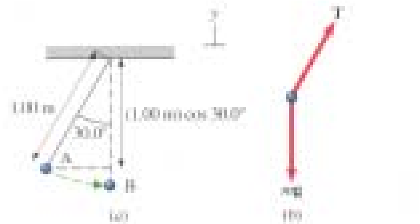


FIGURE 7-13 (*a*) A small ball is hung from the ceiling by a light string. (*b*) The forces on the ball.

EXAMPLE 7-7 Motion of a Body in a Dense Gas

A massless spring with force constant $k = 10$ N/m hangs vertically in a container of dense gas. (See Fig. 7-14*a*.) A small body of mass $m = 0.10$ kg is attached to the end of the spring and released. The body drops a distance of 19 cm before starting back upward. Calculate the work done on the body by the resistive (frictional) force of the gas between the point A where the body is released and the point B where it stops and starts back upward.

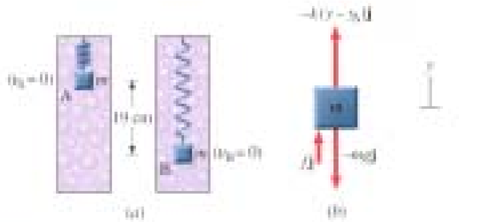
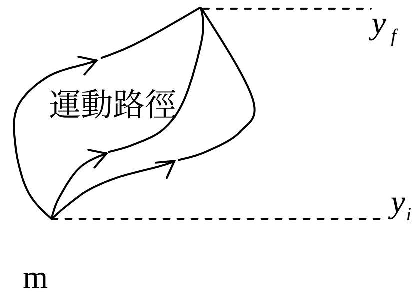


FIGURE 7-14 (*a*) A small body at the end of a spring in a dense gas. (*b*) The free-body diagram of the body.

§7-5　保守力和位能(Conservative forces and potential energy)

△保守力之定義:

若有一力作用於一物體上，並使此物體由某一位置移動至另一位置，若此力對該物體所作的功與物體之運動路徑無關，而僅與物體之初位置 y_i 和末位置 y_f 有關，則此力稱為保守力。換句話說，若 $\oint \vec{F}\cdot d\vec{S}=0$，則 $\vec{F}$ 為保守力。



($\oint \vec{F}\cdot d\vec{S}$ 表力沿著一封閉路徑積分)。

例如:彈力 $\vec{F}=-K\vec{X}$，其所作的功 $W_S=\frac{1}{2}KX_i^2-\frac{1}{2}KX_f^2$ (公式 7-8)，僅與初，末有關，故彈力為一保守力。

例如:地球之重力 $\vec{F}_g=m\vec{g}$，其所作的功 $W_g=mg(y_i-y_f)$ (公式 7-9)，故重力亦為一保守力。

例如:萬有引力 $\vec{F}=-\frac{GMm}{r^2}\hat{r}$，其所作的功 $W=\frac{GMm}{r_f}-\frac{GMm}{r_i}$ (公式 7-10)，故萬有引力為一保守力。

例如:摩擦力 $\vec{F}_K$ 所作的功與運動路徑有關，故摩擦力並非保守力。

△位能之定義:

(Ⅰ)由公式 7-8 知 $W_{S_{IF}}=\frac{1}{2}KX_i^2-\frac{1}{2}KX_f^2$，若定義彈力位能 $U_s=\frac{1}{2}KX^2$，則 $W_{S_{if}}=U_{s_i}-U_{s_f}=-\Delta U_s$。

(Ⅱ)由公式 7-9 知 $W_{gif}=mg(y_i-y_f)$，若定義地表面上之重力位能為 0,

即在高 h 處之$U_g = mgh$，則$W_{gif} = mgh_i - mgh_f = U_{gi} - U_{gh} = -\Delta U$

(Ⅲ)由公式 7-10 知$W_{if} = \frac{GMm}{r_f} - \frac{GMm}{r_i}$，若定義無窮遠處之位能為 0，即 $r_f \to \infty$ 時$U_f = 0$，則某一處之位能$U = -\frac{GMm}{r}$，故 $W_{if} = U_i - U_f = -\Delta U$，故綜合(Ⅰ)(Ⅱ)(Ⅲ)知保守力所作之功 $W_c = -\Delta U$。

§7-6　保守力與位能之關係:

△由上節知$W_C = -\Delta U$

$\therefore$對於一維而言$W_C = \int_{x_i}^{x_f} F(X)dx = -\Delta U$

$\therefore \Delta U = -\int_{x_i}^{x_f} F(x)dx$

若選擇X_i為參考點，且$X_i = 0$時$U_i = 0$

$\Rightarrow U(x) = -\int_0^x F(x)dx$

§7-7　機械能守能(Conservation of machanical energy)

△由 7-4 節知$W_{if} = T_f - T_i$　(在保守力及非保守力情形)

由 7-5 節知$W_{if} = U_i - U_f$　(在保守力情形下)

故可知$T_f - T_i = U_i - U_f$　(在保守力情形下)

$$\Rightarrow T_i + U_i = T_f + U_f$$

若定義機械能$E = T + U$

$\Rightarrow E_i = E_f$此式稱為機械能守恆(在保守力情形下)。

△由上面推導可知，在保守力作用下，機械能守恆，即$E=$定值。

$$\therefore \frac{dE}{dt}=0 \Rightarrow \frac{d(T+U)}{dt}=0 \Rightarrow \frac{dT}{dt}+\frac{dU}{dt}=0$$

$$\Rightarrow \frac{d}{dt}(\frac{1}{2}mV^2)+\frac{dU}{dt}=0 \Rightarrow m\vec{V}\cdot\frac{d\vec{V}}{dt}+\frac{dU}{dt}=0$$

$$\Rightarrow m\vec{V}\cdot\vec{a}+\frac{dU}{dt}=0 \Rightarrow F_X V_X+\frac{dU}{dx}\frac{dx}{dt}=0(\text{在一維情形})$$

$$\Rightarrow F_X V_X+\frac{dU}{dx}V_X=0 \Rightarrow F_X=-\frac{dU}{dx}$$

即若F_X為保守力，則$F_X=-\dfrac{dU}{dx}$

對於三維情形

$$\vec{F}=-\frac{\partial V(r)}{\partial r} \Rightarrow F_X\vec{i}+F_y\vec{j}+F_Z\vec{k}=-\frac{\partial U(r)}{\partial X}\vec{i}-\frac{\partial U(r)}{\partial y}\vec{j}-\frac{\partial U(r)}{\partial Z}\vec{k}$$

§7-8　機械能守值定律的應用

△到目前為止，我們所介紹過的動能僅有移動動能$T=\dfrac{1}{2}mV^2$;所介紹過的位能有兩種;一為重力位能 mgh 或$-\dfrac{GMm}{r}$，另一為彈力性能$\dfrac{1}{2}KX^2$。故我們由機械能守值$E_i=E_f$知

$$T_i+U_i=T_f+U_f$$

$$\Rightarrow \frac{1}{2}mV_i^2+mgh_i+\frac{1}{2}Kx_i^2=\frac{1}{2}mV_f^2+mgh_f+\frac{1}{2}KX_f^2$$

△例題 7-9，7-10，7-12。

EXAMPLE 7-9 ENERGY CONSERVATION FOR A BLOCK AT THE END OF A VERTICAL SPRING

A massless spring with force constant $k = 40$ N/m hangs vertically from the ceiling. A 0.20-kg block is attached to the end of the spring and released. (See Fig. 7-21.) (*a*) What is the maximum extension of the spring? (*b*) What is the block's speed when the spring is extended 4.0 cm?

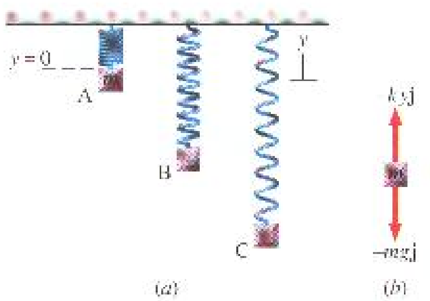


FIGURE 7-21 (*a*) A block is placed on a spring at A and falls to C before rising. Point B is 4.0 cm below point A. (*b*) The forces on the block.

EXAMPLE 7-10 ESCAPE SPEED OF A SPACE CAPSULE

A space capsule of mass m is launched from the surface of the earth with a speed v_E. (*a*) If air friction can be ignored, what is the speed of the capsule as a function of height h above the earth's surface? (*b*) What minimum speed must the capsule have at the surface of the earth if it is to escape from the earth (i.e., approach $h = \infty$)?

EXAMPLE 7-12 ENERGY CONSERVATION FOR A VARIABLE FORCE

A particle of mass 4.0 kg is constrained to move along the x axis under a single force $F(x) = -cx^3$, where $c = 8.0$ N/m^3. The particle's speed at A, where $x_A = 1.0$ m, is 6.0 m/s. What is its speed at B, where $x_B = -2.0$ m?

§有非保守力存在時之機械能

△由 7-4 功能定理知 $W_C + W_{nC} = T_f - T_i$

由 7-5 知 $W_C = U_i - U_f$

$\Rightarrow W_{nc} = T_f - T_i - (U_i - U_f)$

$\Rightarrow W_{nc} = (T_f + U_f) - (T_i + U_i)$

$= E_f - E_i$ (E 表機械能)

$\Rightarrow W_{nc}+T_i+U_i=T_f+U_f$ (此式稱為廣義之能量守恆)

$$\Rightarrow W_{nc}+\frac{1}{2}mV_i^2+mgh_i+\frac{1}{2}KX_i^2=\frac{1}{2}mV_f^2+mgh_f+\frac{1}{2}KX_f^2$$

△例題 7-13

EXAMPLE 7-13 A CART AT THE END OF A COMPRESSED SPRING

Figure 7-22 shows a small cart of mass 2.0 kg that is pressed against a massless spring with force constant $k=2.0\times10^3$ N/m. The spring is compressed 0.30 m and then released. (*a*) When the spring returns to its relaxed length, what is the speed of the cart? Assume that there is no friction. (*b*) If a 40-N frictional force acts on the cart while it is in contact with the spring, what is the cart's speed when the spring returns to its relaxed length?

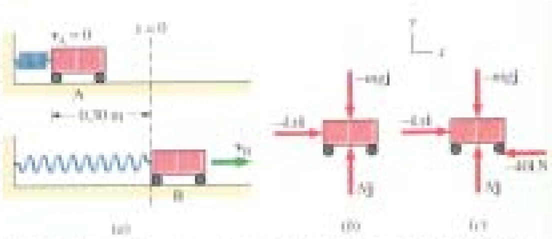


FIGURE 7-22 (*a*) A spring and cart. (*b*) The free-body diagram of the cart with no friction. (*c*) The free-body diagram of the cart in the presence of a 40-N frictional force.

§7-10 功率(power)

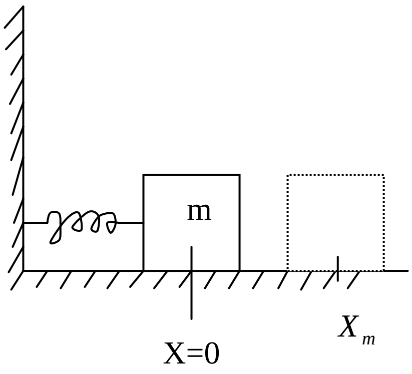
△若有一外力作用於一物體，且在 Δt 時間內對此物體做功 W，則定義平均功率 $\bar{P}\equiv\frac{\Delta W}{\Delta t}$。當 $\Delta t\to 0$ 時，平均功率之極限值稱為瞬時功率(instantaneous power)P，簡稱功率。故功率 $P\equiv\lim_{\Delta t\to 0}\frac{\Delta W}{\Delta t}=\frac{dW}{dt}$，又因為 $dW=\vec{F}\cdot d\vec{S}$，$\therefore P=\frac{dW}{dt}=\frac{d(\vec{F}\cdot d\vec{S})}{dt}$，當 $\vec{F}=$常數，$\Rightarrow P=\vec{F}\cdot\frac{d\vec{S}}{dt}=\vec{F}\cdot\vec{V}$

△功率之 SI 單位為 J/S 或稱為 Watt，但此單位太小，所以我們常用馬力(horse power)來作為功率之單位。$1hp=550\,ft\cdot lb/s=746W$。

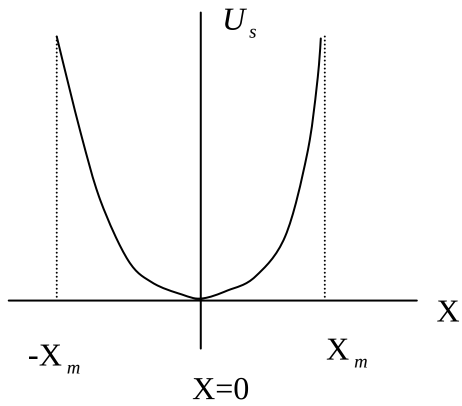
△1 仟瓦一小時(1kwh)=$10^3 w\times3600s=3.6\times10^6 ws=3.6\times10^6 J=1$度(家庭用電常用之單位)。故可知仟瓦一小時為能量之單位，而非功率之單位。

§7-11(補充)

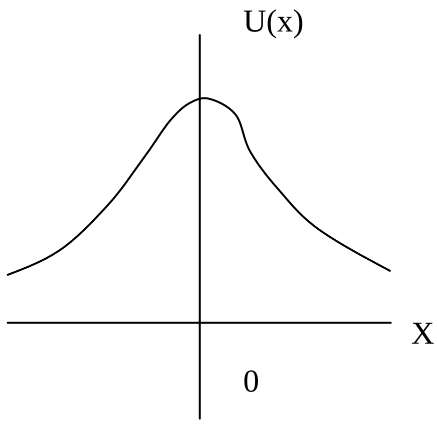
△右圖為一”物體--彈簧”系統的位能圖。



由圖中可知當 $x>0$ 時，$F_x=-\frac{dU}{dx}<0$，當 $x<0$ 時，$F_x=-\frac{dU}{dx}>0$，故可知力永遠與位移方向相反。即質量 m 之物體不管向左或向右移動，物體皆會受到一向 x=0 方向的力之作用，而 x=0 乃位能為極小值之位置。綜合以上討論可知，若 U(x)有極小值，則 U(x)極小值處所對應的位置即為穩定平衡的位置。

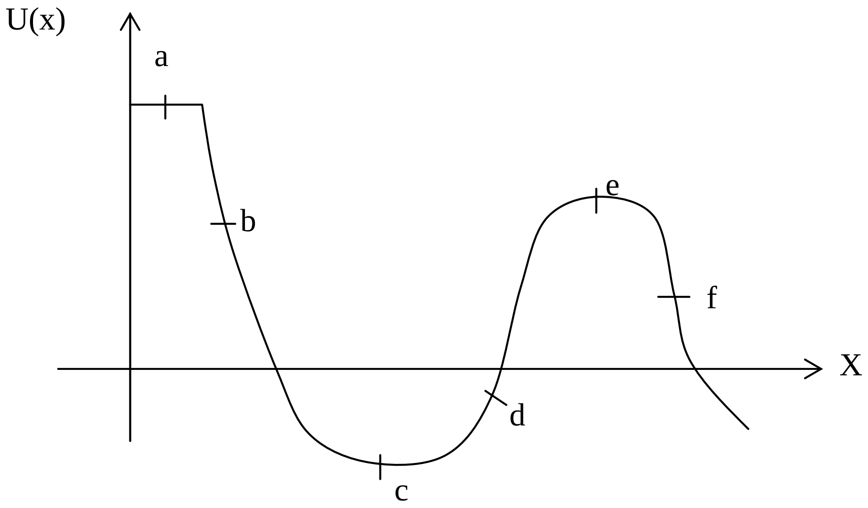


△右圖為 U(x)對 x 的關係圖，由圖中可知，當 $x>0$ 時，$F_X=-\frac{dU}{dx}>0$，且當 $x<0$ 時，$F_X=-\frac{dU}{dx}<0$，故可知力永遠與位移方向相同。即物體不管向右或向左運動，該物體皆會遠離 x=0 之位置。由以上討論可知，若 U(x)有極大值，則 U(x)為極大值處所對應的位置即是不穩定平衡之位置。



△若 U(x)與 x 無關，即 U(x)=constant 時，則 $F_X=-\frac{dU}{dx}=0$，此乃表示 x 不管何值，其物體所受的力皆為零。所以在 U(x)=constant 處，其所對應的位置皆為隨遇平衡(neutral equilibrium)位置。

△綜合以上討論我們可找出下圖位能對 x 的關係圖中, 那些位置為穩定平衡位置, 不穩定平衡位置, 和隨遇平衡位置?



a 為隨遇平衡位置

c 為穩定平衡位置

e 為不穩定平衡位置