機械波

繩波(Wave on a String)

繩波的速度

A. 單位因次分析（Dimensional Analysis）

和波速有關的物理量有繩子單位長度的質量m及繩上之張力T

$$u(T, m)$$

$$\left[\frac{\mathrm{L}}{\mathrm{T}}\right]\left(\left[\frac{\mathrm{ML}}{\mathrm{T}^2}\right],\left[\frac{\mathrm{M}}{\mathrm{L}}\right]\right) \Longleftarrow \text{單位因次}$$

唯一可能

$$u = C\sqrt{\frac{T}{m}}$$

其中C為一無單位之常數。

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B. 力學分析

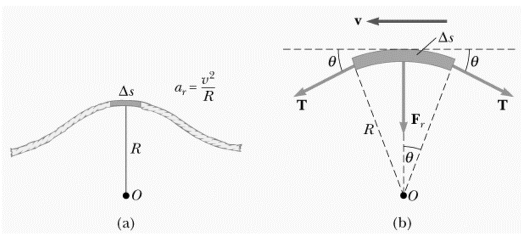
考慮一速度為u 的傳遞波形。由一和繩波速度相同之慣性座標系上觀察，波形靜止不動，繩子本身則相對於波形以v的速度移動。繩子通過波形的頂端部分Δs，可以用半徑為R之圓周運動來做近似。

其所受向心力可寫為 $F = 2T\sin q \approx T(2q) = T\dfrac{\Delta s}{R}$ **當**$q \to 0$

Δs**段之質量為** $\Delta m = m\Delta s$ **加速度為** $a = \dfrac{u^2}{R}$

$$F = ma = m\Delta s\frac{u^2}{R} = T\frac{\Delta s}{R}$$

$$u = \sqrt{\frac{T}{m}}$$



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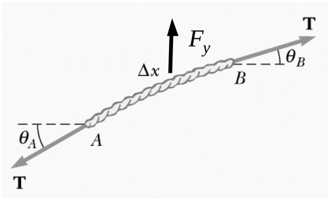
繩波的波動方程式

對於繩波，$f(x,t)=y(x,t)$**，振動之位移**Δy**和行進方向**x**垂直，為一橫波**
(transverse wave)**，由波動方程式可得：**

$$\boxed{\frac{\partial^2 y(x,t)}{\partial t^2}} = u^2 \frac{\partial^2 y(x,t)}{\partial x^2} = \frac{T}{m}\boxed{\frac{\partial^2 y(x,t)}{\partial x^2}}$$

y**方向之加速度(固定**x**)** a_y

曲率(固定t**)**



y**方向受力大小和曲率有關也和張力大小有關，即**

$$F_y \propto T\frac{\partial^2 y(x,t)}{\partial x^2}$$

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又

$$a_y \propto F_y,\ \ a_y \propto \frac{1}{m} \Rightarrow a_y \propto \frac{F_y}{m}$$

即

$$\frac{\partial^2 y(x,t)}{\partial t^2} \propto \frac{T}{m}\frac{\partial^2 y(x,t)}{\partial x^2}$$

＊對於其他介質

$\frac{\partial^2 f}{\partial t^2}$ **可以為某種加速度**

$\frac{\partial^2 f}{\partial x^2}$ **可以為偏離平衡狀態之形變**

$\frac{\partial^2 f}{\partial x^2}\times$**(彈性係數)=恢復力=** $\frac{\partial^2 f}{\partial t^2}\times$ **(慣性項)**

$$\boxed{u^2 = \frac{\text{彈性係數}}{\text{慣性項}}}$$

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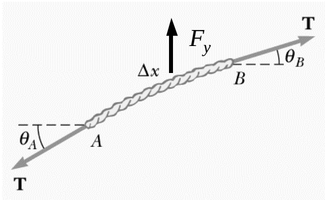
線性波方程式的推導

$$\sum F_y = T\sin q_B - T\sin q_A = T(\sin q_B - \sin q_A)$$

$$\sin q \approx q \approx \tan q \qquad \textbf{if} \qquad q \to 0$$

$$\sum F_y \approx T(\tan q_B - \tan q_A) \approx T\left[\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A\right]$$



又

$$\sum F_y \approx ma_y \approx m\Delta x\left(\frac{\partial^2 y}{\partial t^2}\right) \approx T\left[\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A\right]$$

$$\frac{m}{T}\left(\frac{\partial^2 y}{\partial t^2}\right) \approx \frac{\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A}{\Delta x} \xrightarrow[\Delta x \to 0]{} \frac{\partial^2 y}{\partial x^2}$$

即

$$\frac{m}{T}\left(\frac{\partial^2 y}{\partial t^2}\right) = \frac{\partial^2 y}{\partial x^2}$$

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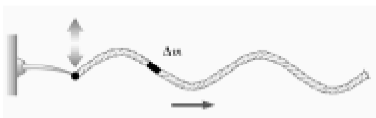
能量的傳遞

A. **位能**



考慮在做簡諧運動的Δm

$$\Delta U = \frac{1}{2}(\Delta m)w^2 y^2 = \frac{1}{2}(m\Delta x)w^2 y^2 \to dU = \frac{1}{2}(mdx)w^2 y^2$$

$$y(x,t) = A\cos(kx - wt)$$

$$dU = \frac{1}{2}mw^2 A^2 \cos^2(kx - wt)dx$$

積分一個波長 $l = 2\pi/k$

$$U_l = \int_0^l dU = \frac{1}{2}mw^2 A^2 \int_0^l \cos^2(kx - wt)dx = \frac{1}{2}mw^2 A^2 \int_0^l \frac{1 + \cos 2(kx - wt)}{2}dx$$

$$= \frac{1}{4}mw^2 A^2 l$$

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B. **動能**

$$\Delta K=\frac{1}{2}(\Delta m)u_y^2=\frac{1}{2}(m\Delta x)\left(\frac{\partial y}{\partial t}\right)^2\rightarrow dK=\frac{1}{2}(mdx)\left(\frac{\partial y}{\partial t}\right)^2$$

$$dK=\frac{1}{2}(mdx)[wA\sin(kx-wt)]^2=\frac{1}{2}mw^2A^2\sin^2(kx-wt)dx$$

積分一個波長 $l=2\pi/k$

$$K_l=\int_0^l dK=\frac{1}{2}mw^2A^2\int_0^l\sin^2(kx-wt)dx=\frac{1}{2}mw^2A^2\int_0^l\frac{1-\cos 2(kx-wt)}{2}dx$$
$$=\frac{1}{4}mw^2A^2 l$$

單位波長之總能 $$E_l=U_l+K_l=\frac{1}{2}mw^2A^2 l$$

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能量的傳輸速率 $$P=\frac{E_l}{\Delta t}=\frac{\frac{1}{2}mw^2A^2 l}{T}=\frac{1}{2}mw^2A^2\left(\frac{l}{T}\right)$$

即 $$P=\frac{1}{2}mw^2A^2u$$

注意 $$P\propto w^2, P\propto^2 A^2, P\propto u$$

P.S.

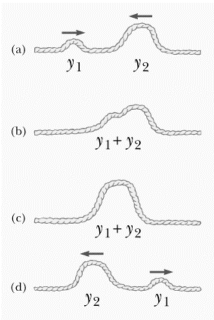
$$P=\frac{dE}{dt}=\frac{dE}{dx}\frac{dx}{dt}=\frac{d(U+K)}{dx}u=\frac{1}{2}mw^2A^2u$$

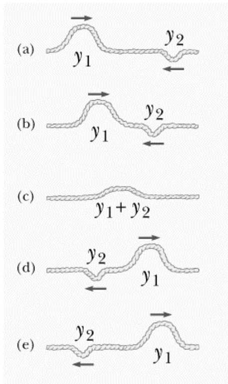
重疊原理(The Principle of Superposition)

線性疊加(Linear Superposition)

A. Overlapping waves algebraically add to produce a **resultant wave**.

B. Overlapping weaves do not in any way alter the travel of each other.





同一介質不同的波相疊加之現象即稱為干涉(interference)**。**

所謂建設性(constructive)**或破壞性**(destructive)**干涉只是相加前的波相位相同**(in phase)**或相反**(reversed phase)**而已。**

數學原理

若 $f_1(x,t)$ **及** $f_2(x,t)$ **均為波方程式** $\dfrac{\partial^2 f(x,t)}{\partial t^2} = v^2 \dfrac{\partial^2 f(x,t)}{\partial x^2}$ **之解，則**

$\Rightarrow af_1(x,t) + bf_2(x,t)$ **亦為** $\dfrac{\partial^2 f(x,t)}{\partial t^2} = v^2 \dfrac{\partial^2 f(x,t)}{\partial x^2}$ **之解**

其中a, b**為任一實數（虛數亦成立）。**

可以利用下面特性證明

$$\frac{\partial^2 af}{\partial t^2} = a\frac{\partial^2 f}{\partial t^2}, \quad \frac{\partial^2 (f_1 + f_2)}{\partial t^2} = \frac{\partial^2 f_1}{\partial t^2} + \frac{\partial^2 f_2}{\partial t^2}$$
$$\frac{\partial^2 af}{\partial x^2} = a\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 (f_1 + f_2)}{\partial x^2} = \frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_2}{\partial x^2}$$

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和向量的特性比較

> **A和B均為空間（向量空間）中之向量**
> ⟹ ***a*A+*b*B=C亦為此空間中之向量**

波的干涉(Interference of Waves)

一維波的干涉

考慮二同向行進、相同波長頻率及振幅、但相位差為*f* 之弦波，

$$y_1(x,t) = y_m \sin(kx - wt)$$
$$y_2(x,t) = y_m \sin(kx - wt + f)$$

他們線性疊加的結果：

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$$y(x,t) = y_1(x,t) + y_2(x,t) = y_m \sin(kx - wt) + y_m \sin(kx - wt + f)$$
$$= 2y_m \cos(\tfrac{1}{2} f)\sin(kx - wt + \tfrac{1}{2} f)$$

（利用 $\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2}$ ）

仍為弦波，振幅和相位差有關。

PHASE DIFFERENCES AND RESULTING INTERFERENCE TYPES[a]

Phase Difference, in			Amplitude of Resultant Wave	Type of Interference
Degrees	Radians	Wavelengths		
0	0	0	$2y_m$	Fully constructive
120	$2\pi/3$	0.33	y_m	Intermediate
180	π	0.50	0	Fully destructive
240	$4\pi/3$	0.67	y_m	Intermediate
360	2π	1.00	$2y_m$	Fully constructive
865	15.1	2.40	$0.60y_m$	Intermediate

註：　距離相差一個波長 l**，相位差**$2p$。
距離相差d**，相位差**$2pd/l$

二維波的干涉