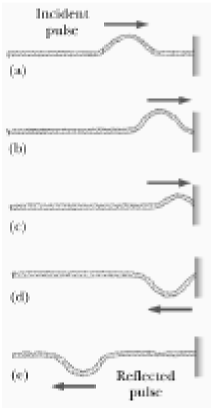
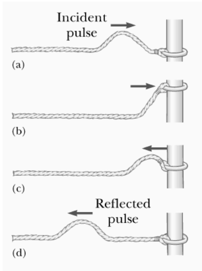
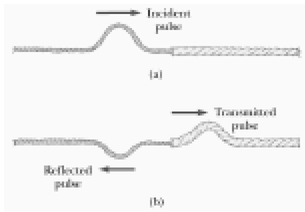
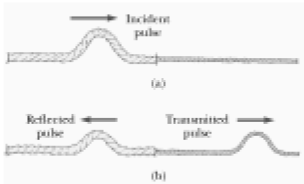
駐波(Standing Waves)

波的反射與穿透(Reflection and Transmission of Waves)









數學模型

入射波 $y(x,t)=A\cos(kx-wt)$

反射波 $y'(x,t)=A'\cos(kx-wt+d_1)$

透射波 $y''(x,t)=A''\cos(k_2x-wt+d_2)$

介質1 **界面** **介質2**

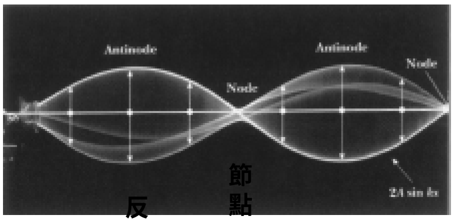
A'、A"、d_1、d_2**由界面的特性決定**

→ Boundary Conditions **邊界條件**

機械波

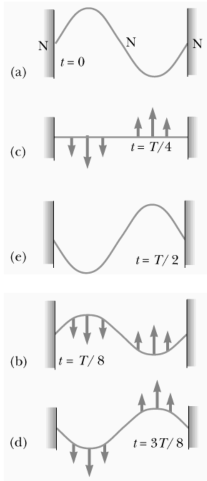
兩端點固定之繩波

—**駐波**(Standing Wave)**之一例**



節點

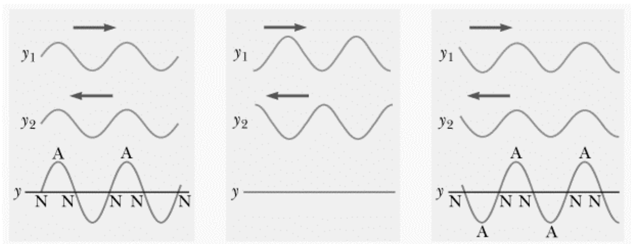
反節點



$$y = (2A\sin kx)\cos wt$$

k**和線長有關**

機械波



(a) $t = 0$ (b) $t = T/4$ (c) $t = T/2$

$$\begin{aligned} y &= (2A\sin kx)\cos wt \\ &= A\sin(kx - wt) + A\sin(kx + wt) \end{aligned}$$

$\longrightarrow$ $\longleftarrow$

利用 $\sin(a \pm b) = \sin a\cos b \pm \cos a\sin b$

機械波

邊界條件與模態(normal mode)

$$y(x,t) = (2A\sin kx)\cos wt$$

Boundary conditions

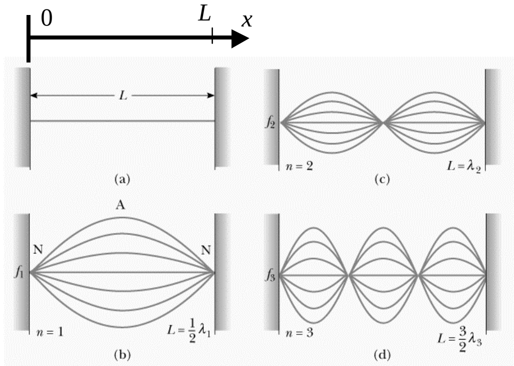
$$y(0,t) = 0$$

$$y(L,t) = 0$$

$$\sin kL = 0$$

$$kL = np = k_n L = \frac{2p}{l_n} L$$

$$n = 1, 2, 3, \ldots$$



每一個n值的振動形式稱做一個振動模態(normal mode)

$$k_n = \frac{np}{L} = \frac{2p}{l_n}, \quad n l_n = 2L, \quad l_n = \frac{2L}{n}, \quad \frac{n}{2} l_n = L$$

機械波

諧波(harmonics)

不同模態的振動頻率並不相同，但呈簡單之有理數比例。

$$w_n = uk_n = n\frac{vp}{L} = nw_o, \quad w_o = \frac{up}{L}$$

$$f_n = \frac{w_n}{2p} = n\frac{u}{2L}$$

其中 $u = \sqrt{\dfrac{T}{m}}$

這些簡單比例的頻率稱為諧波。n愈高之諧波，頻率愈高。

$$y_n(x,t) = 2A_n \sin k_n x \cos w_n t = 2A_n \sin\frac{np}{L} x \cos w_n t$$

不同的模態或諧波可以同時存在同一條振動的紘上且不互相影響，由線性疊加可得合成之波形。

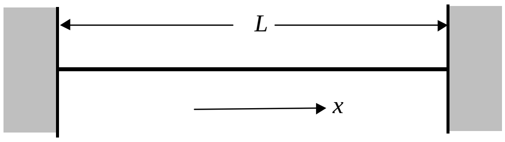
$$y(x,t) = \sum_n y_n(x,t)$$

也可以倒過來做，任何波形可以分解為許多不同模態或諧波之線性疊加。

$\longrightarrow$ **傅利葉級數展開**

機械波

數學處理方法



已知

$m, T \Rightarrow u = \sqrt{\dfrac{T}{m}}$

x=0 $\qquad$ x=L

$y(0,t)$=0 $\quad$ Boundary Conditions (B.C.'s) $\quad$ $y(L,t)$=0

$y(x,t)$ **必須合乎波的方程式** $\dfrac{\partial^2 y(x,t)}{\partial t^2} = u^2 \dfrac{\partial^2 y(x,t)}{\partial x^2}$ **及**B.C.'s

先不考慮B.C.'s**，**$y(x,t)$**之一般解可寫為**

$$y(x,t) = y^+(kx - wt) + y^-(kx + wt), \text{ where } w = uk$$

若只考慮兩傳遞方向相反的弦波(sinusoidal waves)

$$y(x,t) = A^+ \cos(kx - wt) + A^- \cos(kx + wt + d)$$

此時k**（或**w**）並無特定之選擇**

機械波

加入B.C.'s

$y(0,t)$=0 for all t,

$$\begin{aligned}
&y(0,t) = A^+ \cos(-wt) + A^- \cos(wt + d) = 0 \\
&\text{We can choose} \\
&A^+ = -A^- = A, d = 0 \\
&\Rightarrow y(x,t) = A[\cos(kx - wt) - \cos(kx + wt)] \\
&\qquad = 2A \sin kx \sin wt
\end{aligned}$$

$y(L,t)$=0 for all t,

$$\begin{aligned}
&y(L,t) = 2A \sin kL \sin wt = 0 \\
&\Rightarrow kL = np \qquad n = 1,2,3\cdots
\end{aligned}$$

$$k_n = \frac{np}{L} = \frac{2p}{l_n}, \quad n l_n = 2L$$

$$w_n = v k_n = n\frac{vp}{L} = n w_o, \quad w_o = \frac{vp}{L}$$

$$f_n = \frac{w_n}{2p} = n\frac{v}{2L}$$

機械波

對於任何n，$y_n(x,t)$均為合適的解，他們的線性疊加也是合適的解，即任一絃上的波可寫為

$$y(x,t)=\sum_n y_n(x,t)=\sum_n 2A_n\sin k_n x\sin w_n t$$

真正的絃上振動可能是許多模態的線性組合（疊加），由起始條件和邊界條件決定。

能量是否在絃上傳遞？

$$y_n(x,t)=A_n[\cos(k_n x-w_n t)-\cos(k_n x+w_n t)]$$

$$\longrightarrow \qquad \longleftarrow$$

$$\left(\frac{dE}{dt}\right)^{+} \qquad \left(\frac{dE}{dt}\right)^{-}$$

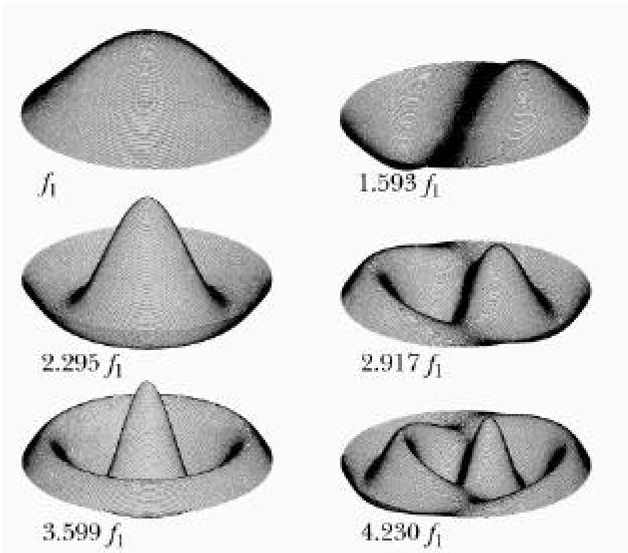
$$\left|\left(\frac{dE}{dt}\right)^{+}\right|=\left|\left(\frac{dE}{dt}\right)^{-}\right|$$

$$\left|\left(\frac{dE}{dt}\right)^{\text{total}}\right|=\left|\left(\frac{dE}{dt}\right)^{+}+\left(\frac{dE}{dt}\right)^{-}\right|=0$$

能量並不傳遞，只留在振動的絃上，故名駐波(Standing Wave)。

機械波

二維的駐波模態



機械波

傳立葉分析(Fourier Analysis)

傳立葉理論(Fourier Theorem)

$f(t)$**是週期性函數**

$$\Leftrightarrow f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nw_0 t + d_n)$$

$$w_0 = \frac{2p}{T}$$

$$A_1 \cos(w_0 t + d_1)$$

$$A_n \cos(n w_0 t + d_n)$$

$$\Leftrightarrow f(t) = \sum_{n=1}^{\infty} A_n \sin(n w_0 t)$$

$$\Leftrightarrow f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n w_0 t)$$

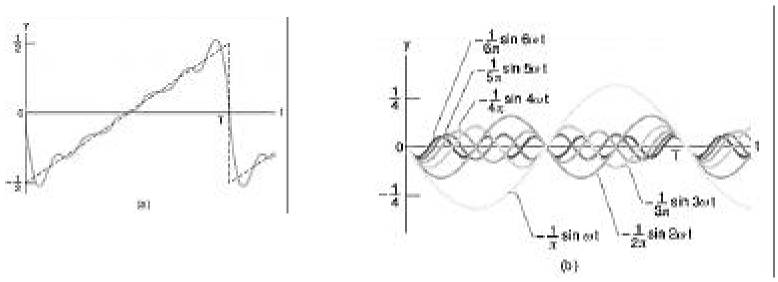
$$w \to k = \frac{2p}{l}$$

機械波

例題

(1) **鋸齒波**(sawtooth wave)

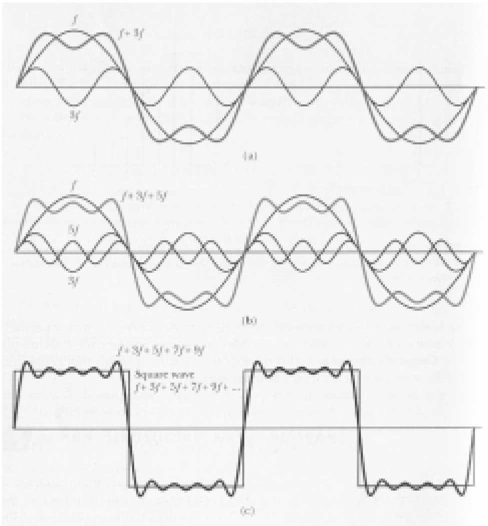
$$y(t) = -\frac{1}{p}\sin wt - \frac{1}{2p}\sin 2wt - \frac{1}{3p}\sin 3wt \cdots\cdots$$



圖中為6**項相加的結果，愈多項則愈和原曲線相像。**

機械波

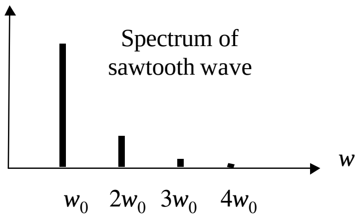
(2) **方波**(square wave)

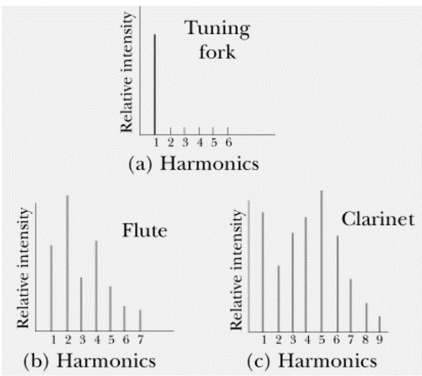


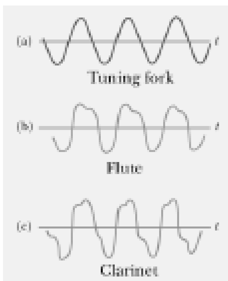
機械波

＊＊頻譜分析 (Fourier Spectrum Analysis) $P(w)$

$P(nw_0) = |A_n|^2$ **或** $P(w) = |A(w)|^2$







共振（Resonance)

當外力或外加之能量的頻率和系統之特徵頻率（包括諧波），系統會產生最大的振盪或反應，此現象稱為共振。

右圖為振幅對應外力之頻率圖。曲線形狀的寬度(Δf)**和振盪的衰減有關。這裡可以用受迫振盪子的數學模型來描述。**$Q = f_0/\Delta f$**。**