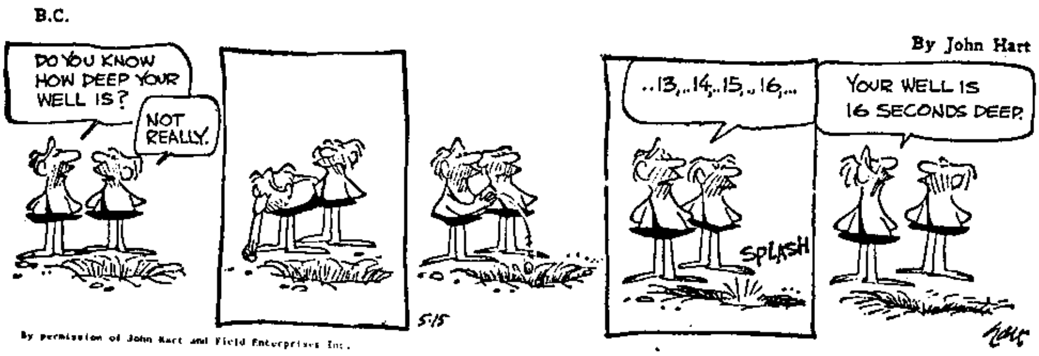
一、運動學(Kinematics)

----運動的描述

(孫允武老師編寫)

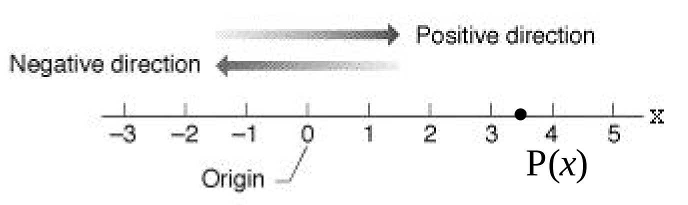
1. **位移與向量？**
2. **速度與加速度**
3. **相對運動**



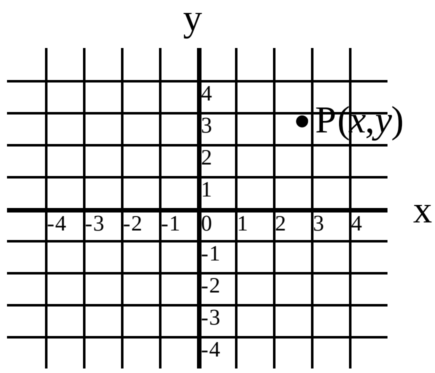
1. 位移與向量(**Displacement** and **Vectors**)

位置(Position)**的描述---座標**(Coordinate)

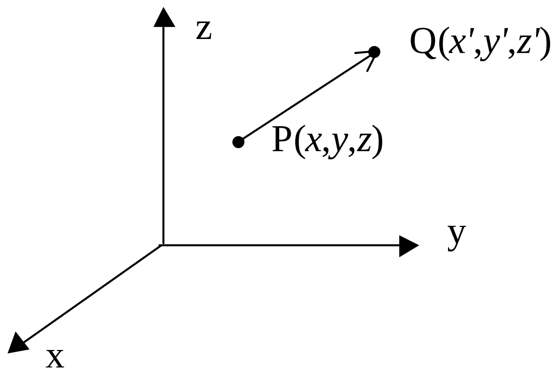
一維系統(one-dimensional system)



二維系統(two-dimensional system)



三維系統(three-dimensional system)



位移---位置的變化量

例如 P→Q **以一向量**(vector)$\overrightarrow{PQ}=\overrightarrow{S_1}$ **表示**

$\overrightarrow{S_1}=(Dx,Dy,Dz)=(x',y',z')-(x,y,z)=(x'-x,y'-y,z'-z)$

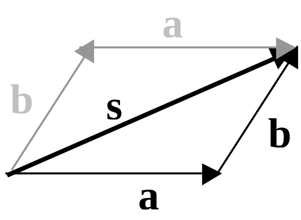
位置---以原點 O 為起點之位移

即 $\vec{P} = \overrightarrow{OP}$

（方便起見，後面之向量以粗體字表示）

- **位移的加成→向量的加法**(addition)

s = **a**+**b**

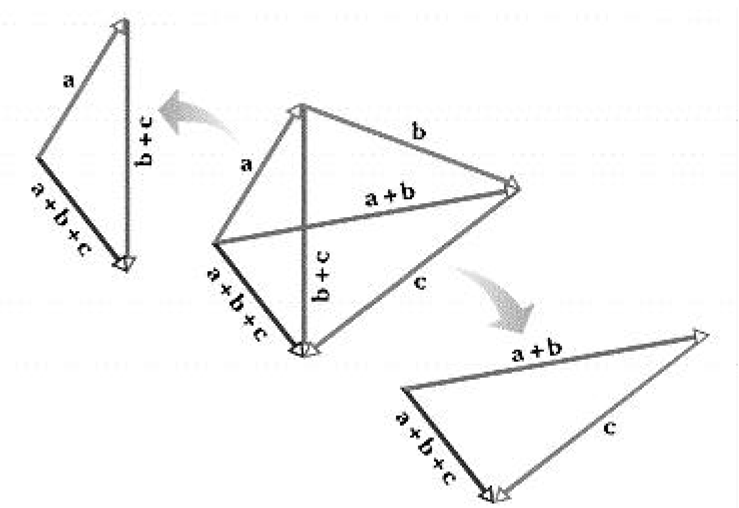


- **加法的交換律**(commutative law)

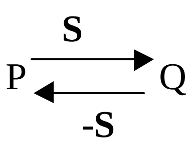
a+**b** = **b**+**a**

- **加法的結合律**(associative law)

(**a**+**b**)+**c** = **a**+(**b**+**c**)



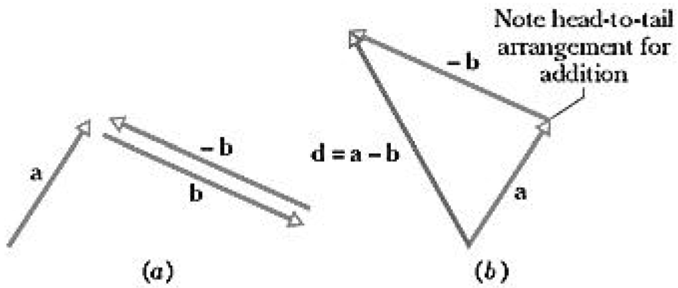
- **位移的反向→向量的加法反元素**



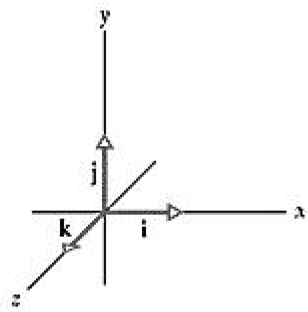
S+(-**S**) = **0** (**零向量，** Null Vector)

- **向量的減法**(Subtration)

  $\mathbf{d} = \mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$



- **單位向量**(Unit Vectors)



$|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$, and $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{i} \cdot \mathbf{k} = 0$ (**互相垂直**)

手寫符號 $\mathbf{i} \rightarrow \hat{\imath}$, $\mathbf{j} \rightarrow \hat{\jmath}$, $\mathbf{k} \rightarrow \hat{k}$

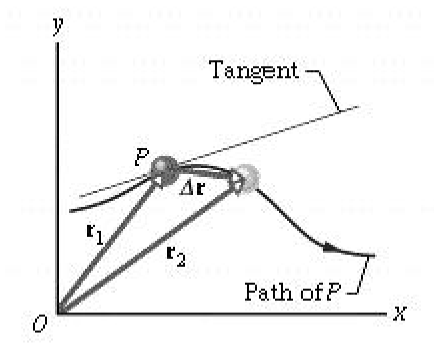
$P(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

2. 速度與加速度(**Velocity** and **Acceleration**)

---對時間的微分與積分

一物體在空間中之運動軌跡或路徑(path)**可用一隨時間改變的位置向量表示，即** $\mathbf{r}(t) = (x(t), y(t), z(t))$。

在時間 t_1 **的位置為** $\mathbf{r}_1$**，在** t_2 **為** $\mathbf{r}_2$。



2.1 速度(velocity)

平均速度(average velocity) $\bar{\mathbf{v}}$

$$\equiv \frac{\Delta \mathbf{r}}{\Delta t} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{t_2 - t_1}$$

$$= \left(\frac{x_2 - x_1}{t_2 - t_1}, \frac{y_2 - y_1}{t_2 - t_1} \right) = \left(\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t} \right)$$

速度本身也是向量，單位是 m/s。

平均速率(average speed) $\bar{v} \equiv |\bar{\mathbf{v}}| = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}$

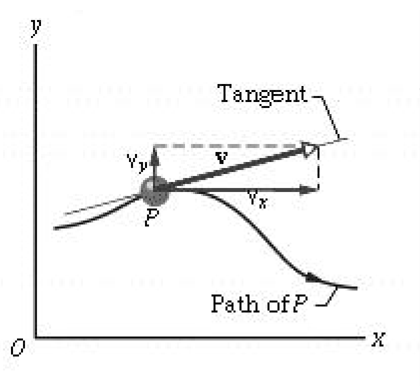
[**瞬時**]**速度**([instantaneous] velocity) $\mathbf{v}$

$$\equiv \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} \left(\equiv \frac{d\mathbf{r}}{dt} \equiv \dot{\mathbf{r}} \right) = \lim_{t_2 \to t_1} \frac{\mathbf{r}_2 - \mathbf{r}_1}{t_2 - t_1} = \lim_{t_2 \to t_1} \left(\frac{x_2 - x_1}{t_2 - t_1}, \frac{y_2 - y_1}{t_2 - t_1} \right)$$

$$= \lim_{\Delta t \to 0} \left(\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t} \right) = \left(\frac{dx}{dt}, \frac{dy}{dt} \right) = (\dot{x}, \dot{y})$$

速度是位移對時間之一次微分。

The instantaneous velocity **v** of a particle is always tangent to the path of the particle.



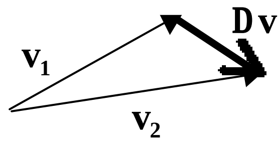
2.2 加速度(acceleration)

平均加速度(average acceleration) $\bar{\mathbf{a}}$

$$\equiv \frac{\Delta \mathbf{v}}{\Delta t} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{t_2 - t_1}$$

$$= \left(\frac{v_{x2} - v_{x1}}{t_2 - t_1}, \frac{v_{y2} - v_{y1}}{t_2 - t_1} \right) = \left(\frac{\Delta v_x}{\Delta t}, \frac{\Delta v_y}{\Delta t} \right)$$

,**加速度本身也是向量，單位是** m/s²。



[瞬時]加速度([instantaneous] acceleration) **a**

$$\equiv \lim_{\text{Ä}t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} \left(\equiv \frac{d\mathbf{v}}{dt} \equiv \dot{\mathbf{v}} \right) = \lim_{t_2 \to t_1} \frac{\mathbf{v}_2 - \mathbf{v}_1}{t_2 - t_1} = \lim_{\text{Ä}t \to 0} \left(\frac{\Delta v_x}{\Delta t}, \frac{\Delta v_y}{\Delta t} \right) = \left(\frac{dv_x}{dt}, \frac{dv_y}{dt} \right) = (\dot{v}_x, \dot{v}_y)$$

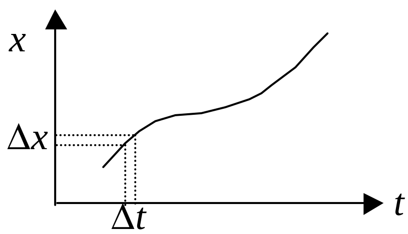
$$= \frac{d}{dt} \left(\frac{d\mathbf{r}}{dt} \right) = \frac{d^2 \mathbf{r}}{dt^2} = \ddot{\mathbf{r}}$$

加速度是速度對時間之一次微分，是位移之二次微分。

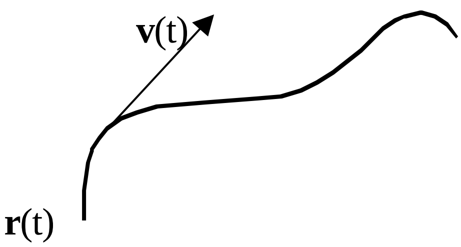
微分之幾何意義及計算原則

(1) **斜率**(slope)

slope=$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$



(2) **切線**(tangent line)



(3) 基本微分計算法則

(1) $f(t)=g(t)+h(t)\Rightarrow \frac{df(t)}{dt}=\frac{dg(t)}{dt}+\frac{dh(t)}{dt}$

(2) $f(t)=g(t)h(t)\Rightarrow \frac{df(t)}{dt}=\frac{dg(t)}{dt}h(t)+g(t)\frac{dh(t)}{dt}$

(3) $f(t)=g(h(t))\Rightarrow \frac{df(t)}{dt}=\frac{dg(h(t))}{dh}\frac{dh(t)}{dt}$ chain rule

(4) $f(t)=qt^{n}\Rightarrow \frac{df(t)}{dt}=qnt^{n-1}$

(5) $\frac{d\sin x}{dx}=\cos x, \frac{d\cos x}{dx}=-\sin x$

**記號 $\frac{df(t)}{dt}=f'(t), \frac{d}{dt}\left[\frac{df(t)}{dt}\right]=\frac{d^2 f(t)}{dt^2}=f''(t)$

2.3 例子

(1) 一維運動(one-dimensional motion)

a. $x(t)=ht+s, v(t)=\frac{dx}{dt}=h$; $s=x(0), v=const.=h$

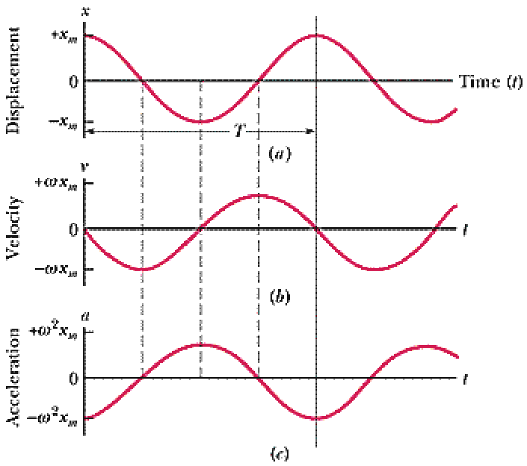
$\Rightarrow x(t)=vt+x(0)$，**等速運動**(constant-velocity motion)

b. $x(t)=ct^2+ht+s, v(t)=\frac{dx}{dt}=2ct+h, a(t)=\frac{dv}{dt}=2c$;

$x(0)=s, v(0)=h, a=const.=2c$

$\Rightarrow x(t)=\frac{a}{2}t^2+v(0)t+x(0)$，**等加速度運動**(constant-acceleration motion)

c. $x(t)=x_m\cos wt, v(t)=-x_m w\sin wt,$
$a(t)=-x_m w^2\cos wt$



週期性運動(periodic motion)

簡諧振盪(simple harmonic oscillation, SHM)

$\frac{d^2x(t)}{dt^2}=-w^2x(t)$ **微分方程式**

(2) 二維及三維運動(two- and three-dimensional motions)

a. $\mathbf{r}(t) = (x(t), y(t)) = (h_x t + s_x, h_y t + s_y)$

$$\mathbf{r}(0) = (s_x, s_y)$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt} \right) = (h_x, h_y) = (v_x, v_y) \quad \text{constant velocity}$$

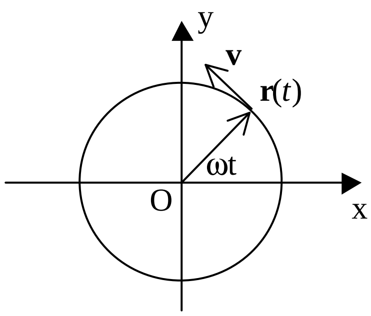
$$\mathbf{r}(t) = (v_x, v_y)t + (s_x, s_y) = t\mathbf{v} + \mathbf{r}(0)$$

b. $\mathbf{r}(t) = \frac{1}{2} t^2 \mathbf{a} + t\mathbf{v}(0) + \mathbf{r}(0)$ constant acceleration

c. **等速率圓周運動**(uniform circular motion)

$$\mathbf{r}(t) = (x(t), y(t)) = (r_0 \cos wt, r_0 \sin wt) = r_0 (\cos wt, \sin wt)$$

$x^2 + y^2 = r_0{}^2$ **路徑或軌跡**(path)



ω **角頻率**(angular frequency) **單位** rad/s

ωT=2π, T=2π/ω **週期**(period)

f=1/T=ω/2π **頻率**(frequency)

單位 1/s(Hz)

$$\mathbf{v}(t) = (v_x(t), v_y(t)) = \frac{d\mathbf{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt} \right) = (-r_0 w \sin wt, r_0 w \cos wt) = r_0 w(-\sin wt, \cos wt)$$

$v_x{}^2 + v_y{}^2 = r_0{}^2 w^2 = v_0{}^2, v_0 = r_0 w$ **速率不變但方向時時在變**

$\mathbf{v}(t) \cdot \mathbf{r}(t) = 0$ **$\mathbf{v}(t)$和 $\mathbf{r}(t)$時時垂直**

$$\mathbf{a}(t) = (a_x(t), a_y(t)) = \frac{d\mathbf{v}}{dt} = \left(\frac{dv_x}{dt}, \frac{dv_y}{dt} \right) = (-r_0 w^2 \cos wt, -r_0 w^2 \sin wt) = r_0 w^2 (-\cos wt, -\sin wt)$$

$a_x{}^2 + a_y{}^2 = r_0{}^2 w^4 = a_0{}^2, a_0 = r_0 w^2$ **大小不變但方向時時在變**

$$\hat{\mathbf{a}}(t) = \frac{\mathbf{a}(t)}{|\mathbf{a}|} = (-\cos wt, -\sin wt) = -\frac{\mathbf{r}(t)}{|\mathbf{r}|} = -\hat{\mathbf{r}}(t)$$

$\hat{\mathbf{v}}(t)\cdot\hat{\mathbf{a}}(t)=0$

等速率圓周運動中，粒子之速度時時和加速度垂直，且加速度方向一直向著圓心。

$$v_0 = r_0\omega \qquad a_0 = r_0\omega^2 = \frac{v_0^{\ 2}}{r_0}$$

r(t)**是下列微分方程式的解**

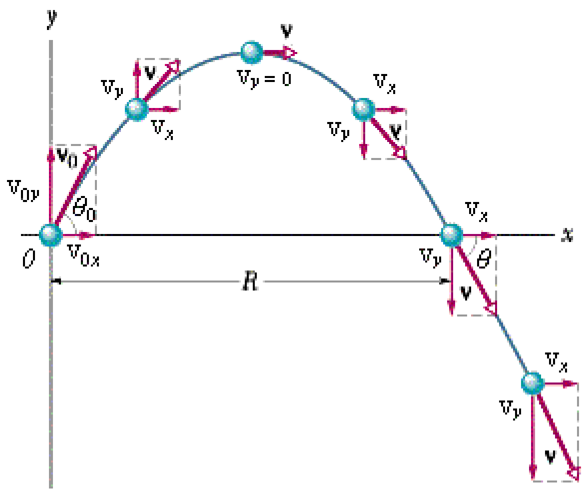
$$\frac{d^2\mathbf{r}(t)}{dt^2} = -\omega^2\mathbf{r}(t),\ \text{即}\ \frac{d^2x(t)}{dt^2} = -\omega^2 x(t),\ \frac{d^2y(t)}{dt^2} = -\omega^2 y(t)$$

d. **螺線運動**(helical motion)

$$\mathbf{r}(t) = (x(t), y(t), z(t)) = (r_0\cos\omega t, r_0\sin\omega t, v_z t)$$

e. **拋物運動**(projectile motion)

→ constant-speed motion



↓ constant-acceleration motion $\qquad \mathbf{g} = -g\hat{\mathbf{y}}$

$$\mathbf{r}(t) = (x(t), y(t)) = (v_{0x}t + x_0, -\frac{1}{2}gt^2 + v_{0y}t + y_0)$$

其中 $v_{0x} = v_0\cos\theta_0, v_{0y} = v_0\sin\theta_0$

$$x(t) = v_0 \cos q_0 t + x_0$$
$$y(t) = -\frac{1}{2} g t^2 + v_0 \sin q_0 t + y_0$$

消去 t 可得軌跡方程式

$$y - y_0 = (\tan q_0)(x - x_0) - \frac{g(x - x_0)^2}{2(v_0 \cos q_0)^2}$$

為一拋物線

例題　**求上述拋物運動之水平射程**(horizontal range)R **及當** R **為最大時之**q_0。

解答

令 y=0，**可解得** x=0(trivial solution)**或** $\frac{2v_0^2}{g}\sin q_0 \cos q_0$

故 $R = \frac{2v_0^2}{g}\sin q_0 \cos q_0$。

利用複角公式可得 $R = \frac{v_0^2}{g}\sin 2q_0$，**又** $\sin 2q_0 \le 1$

故 R **之最大值為** v_0^2/g，**此時** $\sin 2q_0 = 1$，**即** $q_0 = 45^\circ$ 。

例題　**求上述拋物運動軌跡之最高點座標**。

解答　**在軌跡之最高點時**，v_y=0。

$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = (v_{0x}, -gt + v_{0y})$，**可得** $t = v_{0y}/g$ **時** v_y=0，**帶入** $\mathbf{r}(t)$**得最高點座標** $\left(x_0 + \frac{v_{0x}v_{0y}}{g}, y_0 + \frac{v_{0y}^2}{2g} \right)$。

另解　**令** $\frac{dy}{dx} = 0$，**得** $x_{\max} = x_0 + \frac{v_0^2 \sin q_0 \cos q_0}{g} = x_0 + \frac{v_{0x}v_{0y}}{g}$，**此時**

$y_{\max} = y_0 + \frac{v_0^2 \sin^2 q_0}{2g} = y_0 + \frac{v_{0y}^2}{2g}$。

2.4 微分之反運算

----積分(Integration)

已知速度 $\mathbf{v}(t)$求位移 $\mathbf{r}(t)$，或已知加速度 $\mathbf{a}(t)$求速度 $\mathbf{v}(t)$。

(1) $\mathbf{v}(t)\rightarrow\mathbf{r}(t)$

$$\frac{d\mathbf{r}}{dt}=\mathbf{v}(t)\Rightarrow d\mathbf{r}=\mathbf{v}dt \qquad (\Delta\mathbf{r}=\mathbf{v}\Delta t,\Delta t\rightarrow 0)$$

我們可將有興趣的時間範圍($t_i\rightarrow t_f$)分為 n 等分，$\Delta t=\dfrac{t_f-t_i}{n}$，在第 k 等分之速度以 $\mathbf{v}(t_k)$做近似，t_k 為該等分中點之時間值。在 t_f 的位置可由下式求出

$$\mathbf{r}(t_f)=\mathbf{r}(t_i)+\lim_{n\rightarrow\infty}\sum_{k=1}^{n}\mathbf{v}(t_k)\Delta t$$

其中$\lim_{n\rightarrow\infty}\sum_{k=1}^{n}\mathbf{v}(t_k)\Delta t$ **可以記做**$\int_{t_i}^{t_f}\mathbf{v}dt$**，即**

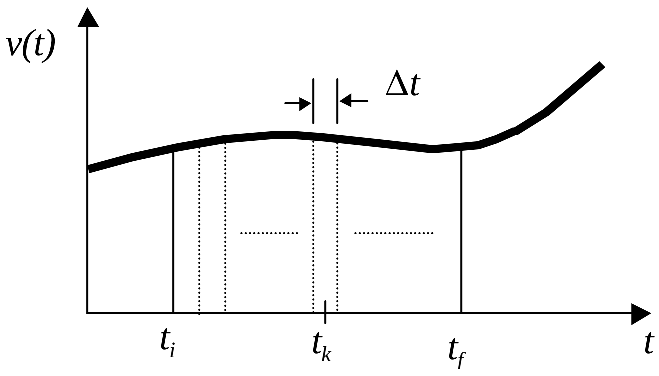
$$\mathbf{r}(t_f)-\mathbf{r}(t_i)=\int_{t_i}^{t_f}\mathbf{v}dt$$

又可寫成各分量之形式：

$$x(t_f)-x(t_i)=\int_{t_i}^{t_f}v_x dt$$

$$y(t_f)-y(t_i)=\int_{t_i}^{t_f}v_y dt$$

$$z(t_f)-z(t_i)=\int_{t_i}^{t_f}v_z dt$$



***積分(integration)——微分(differentiation)的反運算

$$\frac{dF(t)}{dt} = f(t) \Leftrightarrow F(t_2) - F(t_1) = \int_{t_1}^{t_2} f(t)dt$$

$dF(t) = f(t)dt$，**左右積分得** $\int_{t=t_1}^{t=t_2} dF(t) = \int_{t_1}^{t_2} f(t)dt$，**其中**

$$\int_{t=t_1}^{t=t_2} dF(t) = F(t)\Big|_{t_1}^{t_2} = F(t_2) - F(t_1)$$

例題

(1) $\int_{x_1}^{x_2} dx = x\Big|_{x_1}^{x_2} = x_2 - x_1$

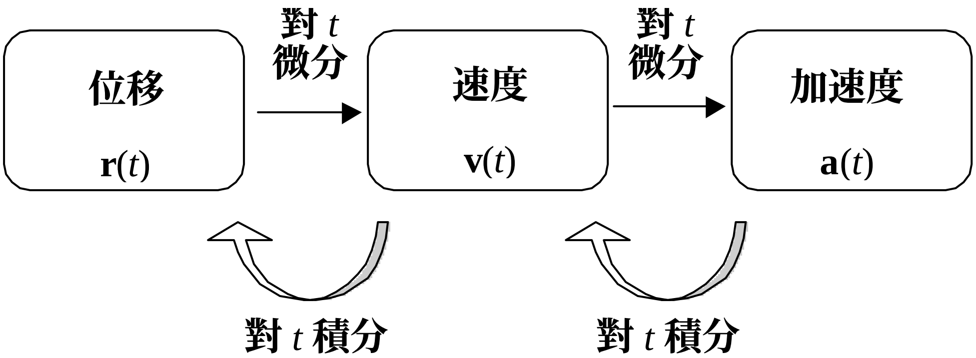
(2) $\int_{x_1}^{x_2} x^n dx = \frac{1}{n+1} x^{n+1}\Big|_{x_1}^{x_2} = \frac{1}{n+1}(x_2^{\ n+1} - x_1^{\ n+1})$

(3) $\int_{x_1}^{x_2} \cos x dx = \sin x\Big|_{x_1}^{x_2} = \sin x_2 - \sin x_1$

(2) $\mathbf{a}(t) \rightarrow \mathbf{v}(t)$

同理 $\frac{d\mathbf{v}}{dt} = \mathbf{a}(t) \Rightarrow d\mathbf{v} = \mathbf{a}dt \qquad (\Delta\mathbf{v} = \mathbf{a}\Delta t, \Delta t \to 0)$

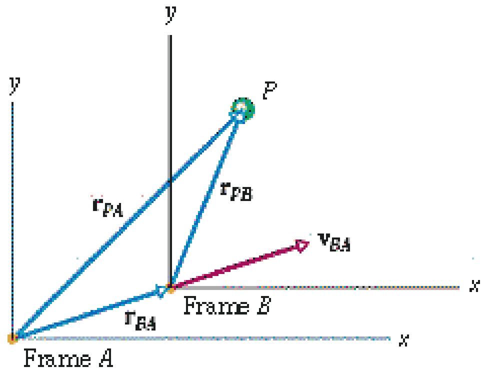
$$\Rightarrow \int_{t=t_i}^{t=t_f} d\mathbf{v} = \mathbf{v}(t_f) - \mathbf{v}(t_i) = \int_{t_i}^{t_f} \mathbf{a}dt$$



3. 相對運動(Relative Motion)

----座標轉換

座標 B 原點對於座標 A 原點之速度為 $\mathbf{v}_{BA}$，只考慮 B 與 A 之對應座標軸互相平行。



$\mathbf{r}_{PA}(t)=\mathbf{r}_{PB}(t)+\mathbf{r}_{BA}(t)$

對 t 微分得(座標 A 與 B 的時間是相同的)

$\frac{d\mathbf{r}_{PA}}{dt}=\frac{d\mathbf{r}_{PB}}{dt}+\frac{d\mathbf{r}_{BA}}{dt}$ 即 $\mathbf{v}_{PA}(t)=\mathbf{v}_{PB}(t)+\mathbf{v}_{BA}(t)$ (速度的〝加法〞)

在對 t 微分可得 $\mathbf{a}_{PA}(t)=\mathbf{a}_{PB}(t)+\mathbf{a}_{BA}(t)$

- 若 $\mathbf{a}_{BA}$=0(即 $\mathbf{v}_{BA}$=const.)則 $\mathbf{a}_{PA}(t)=\mathbf{a}_{PB}(t)$

  在互相做等速度運動的座標系統觀察到同一質點運動之加速度相同。

- 若 $\mathbf{v}_{BA}$=0(即 $\mathbf{r}_{BA}$=const.)則 $\mathbf{v}_{PA}(t)=\mathbf{v}_{PB}(t)$

在純運動學觀點，這些座標之選擇除數學描述方便與否外，並不具任何意義。但到牛頓力學中則有很大的差別。

**** 當相對速度很低之座標(考慮一維)間 $v_{PA}=v_{PB}+v_{BA}$

當速度很快(差不多光速的數量級)時，速度之轉換則非線性相加而已：$v_{PA}=\dfrac{v_{PB}+v_{BA}}{1+v_{PB}v_{BA}/c^2}$ (可由狹義相對論 special relativity 導得)

例題 $v_{PB}=v_{BA}=0.65c$，試計算 v_{PA}。

解答 若依一般之速度加法則得

$v_{PA}=v_{PB}+v_{BA}=1.30c>c$ 違反我們所觀察到的自然界現象。

若依相對論的結果

$$v_{PA}=\frac{v_{PB}+v_{BA}}{1+v_{PB}v_{BA}/c^2}=\frac{0.65c+0.65c}{1+(0.65c)(0.65c)/c^2}=0.91c<c$$ 。

照相對論的結果，我們無法使物質的運動速度超越光速。

c=299,792,458 m/s

討論問題

Q 4 -9

Figure 4-25 shows three paths for a kicked football. Ignoring the effects of air on the flight, rank the paths according to
(a) time of flight,
(b) initial vertical velocity component,
(c) initial horizontal velocity component, and
(d) initial speed.
Place the greatest first in each part.

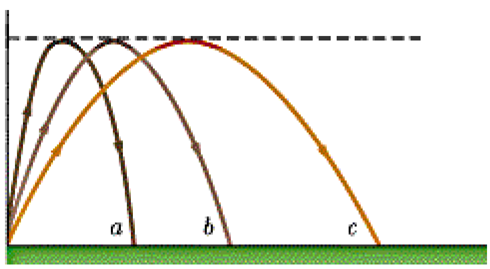


FIGURE 4-25

Q4-12

(a) Is it possible to be accelerating while traveling at constant speed ?
Is it possible to round a curve with (b) zero acceleration and
(c) a constant magnitude of acceleration ?

Q4-17

Figure 4-27 shows one of four star cruisers that are in a race. As each cruiser passes the starting line, a shuttle craft leaves the cruiser and races toward the finish line. You are the judge and are stationary relative to the start and finish lines. The speeds v_c of the four cruisers relative to you and the speeds v_s of the shuttles relative to their corresponding cruisers are, respectively:

(1) 0.70*c*, 0.40*c*; (2) 0.40*c*, 0.70*c*; (3) 0.20*c*, 0.90*c*; (4) 0.50*c*, 0.60*c*.

Without written calculation, determine

(a) which shuttle wins the race and (b) which is last.

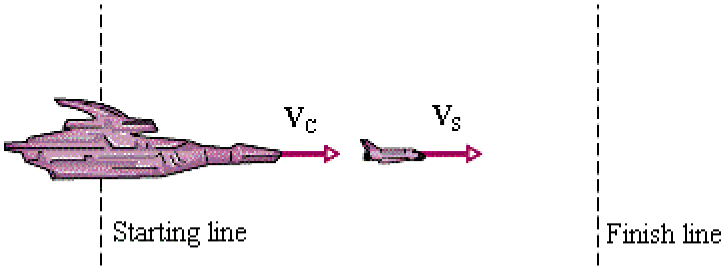


FIGURE 4-27