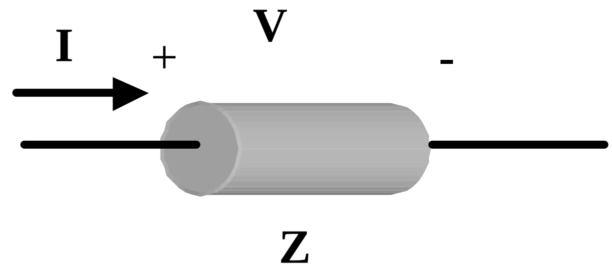
交流訊號的功率

考慮一AC電流通過某阻抗為Z的元件



$$\mathbf{Z} = |Z|\, e^{jf}$$

$$i(t) = I_{\max} \cos wt$$

$$u(t) = V_{\max} \cos(wt + f)$$

$$\mathrm{P}(t) = i(t)u(t) = I_{\max} V_{\max} \cos wt \cos(wt + f)$$

P(t)為瞬時功率(instantaneous power)

定義平均功率(average power) $\mathrm{P}_{\mathrm{av}}(t)$

$$\mathrm{P}_{\mathrm{av}}(t) = \lim_{t \to \infty} \frac{1}{t} \int_0^t i(t)u(t)\,dt$$

$$= \frac{1}{T} \int_0^T i(t)u(t)\,dt \quad \text{for periodic signal with period of } T$$

$$P_{av} = \frac{1}{T}\int_0^T I_{max}V_{max}\cos wt\cos(wt+f)dt$$

$$= \frac{I_{max}V_{max}}{T}\int_0^T \cos wt[\cos wt\cos f - \sin wt\sin f]dt$$

$$= \frac{I_{max}V_{max}}{T}\int_0^T [\cos^2 wt\cos f - \cos wt\sin wt\sin f]dt$$

$$= \frac{I_{max}V_{max}}{2}\cos f$$

又 $V_{max} = I_{max}|Z|$

$$P_{av} = \frac{I_{max}V_{max}}{2}\cos f = \frac{I_{max}^2|Z|}{2}\cos f = \frac{V_{max}^2}{2|Z|}\cos f$$

和直流的情形比較 $P_{av} = IV = I^2R = \dfrac{V^2}{R}$

我們稱cosϕ**為功率因子**(power factor)，ϕ**恰為元件或電路阻抗的相角。**

對於純電容或電感的元件，ϕ為±π/2，cosϕ=0，平均消耗功率為零。電容與電感為能量儲存元件，不消耗能量。電阻則為能量消耗元件， cosϕ=1。

對電阻而言

$$P_{avR} = \frac{I_{max}V_{max}}{2} = \frac{I_{max}^2 R}{2} = \frac{V_{max}^2}{2R}$$

我們可以定義一方便的平均值使得

$$I_{av}^2 = \frac{I_{max}^2}{2} \qquad V_{av}^2 = \frac{V_{max}^2}{2}$$

即 $I_{av} = \frac{I_{max}}{\sqrt{2}} = 0.707 I_{max} \qquad V_{av} = \frac{V_{max}}{\sqrt{2}} = 0.707 V_{max}$ **這個平均恰好是所謂的方均根(rms)值**

$$P_{av} = I_{av}V_{av}\cos f = I_{av}^2 |Z| \cos f = \frac{V_{av}^2}{|Z|}\cos f$$

$$= I_{rms}V_{rms}\cos f = I_{rms}^2 |Z| \cos f = \frac{V_{rms}^2}{|Z|}\cos f$$

或用複數表示

$$P_{av} = \mathrm{Re}[\mathbf{I}\mathbf{V}^*] = \mathrm{Re}[\mathbf{I}^*\mathbf{V}]$$

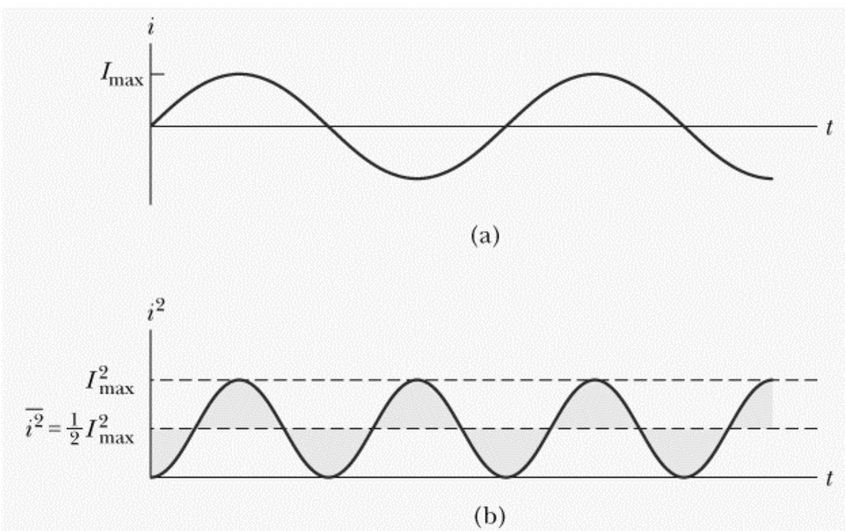
$$= I_{rms}V_{rms}\cos f$$

方均根值(root mean square, rms)

他的基本定義：$f_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T [\,f\,(t)]^2\,dt}$

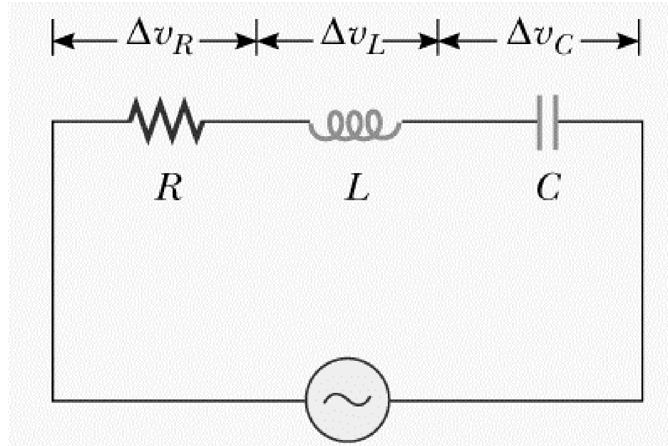
假如$f(t)$是平均為0的弦波函數，即 $f\,(t) = f_{\max}\cos(wt + q\,)$

$$f_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T [\,f_{\max}\cos(wt + q\,)]^2\,dt} = \frac{|\,f_{\max}\,|}{\sqrt{2}}$$



假如訊號不是弦波或平均值不是0，則上式便不成立。

*RLC*串聯共振電路與受迫振盪子 (RLC Series Circuit and Forced Oscillator)



假設電壓訊號是 $u(t) = V_{\max} \cos wt$

*RLC*串聯的阻抗是

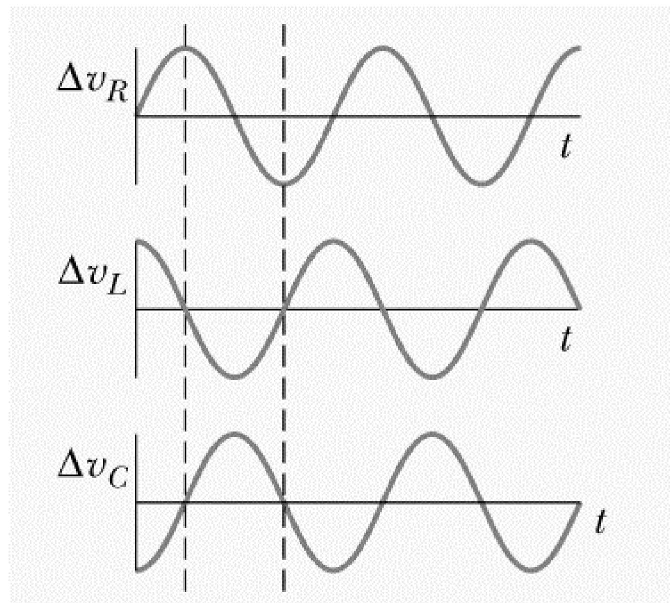
$$\mathbf{Z} = R + jwL + \frac{1}{jwC} = R + j\left(wL - \frac{1}{wC}\right) = R + jX$$

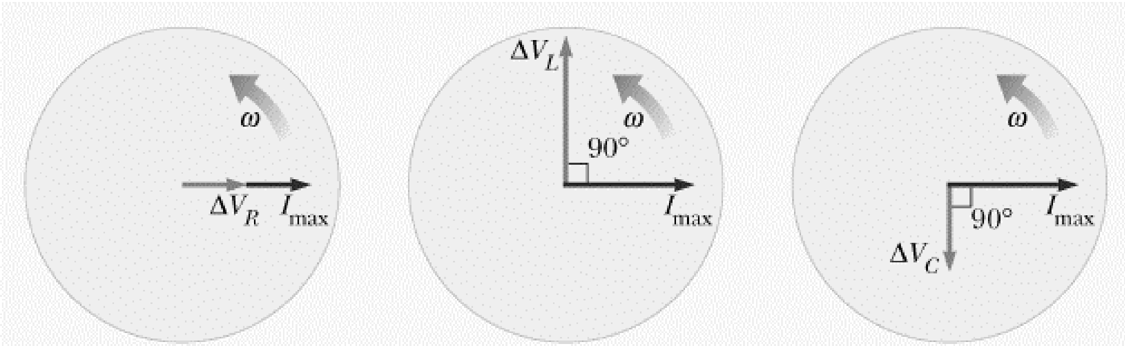
$$|\mathbf{Z}| = \sqrt{R^2 + \left(wL - \frac{1}{wC}\right)^2} \qquad \arg(\mathbf{Z}) = \tan^{-1}\frac{\left(wL - \frac{1}{wC}\right)}{R}$$

低頻時，$w < \frac{1}{\sqrt{LC}}$ $X<0$，主要是電容性的電抗。

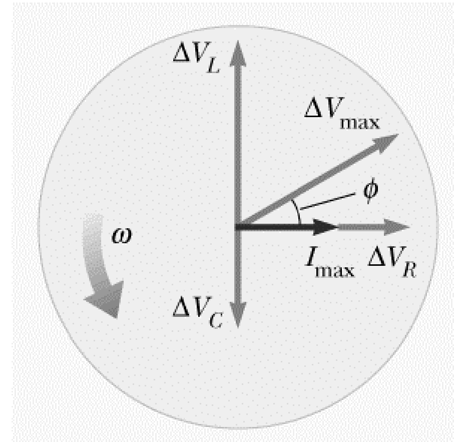
高頻時，$w > \frac{1}{\sqrt{LC}}$ $X>0$，主要是電感性的電抗。

當 $w = \frac{1}{\sqrt{LC}}$ $X=0$ 此時只剩電阻的阻抗，阻抗達最小值，電流最大，稱做共振(resonance)。

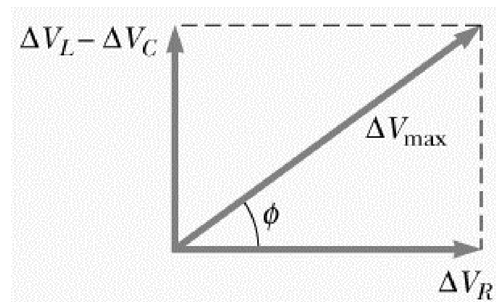




(a) Resistor (b) Inductor (c) Capacitor



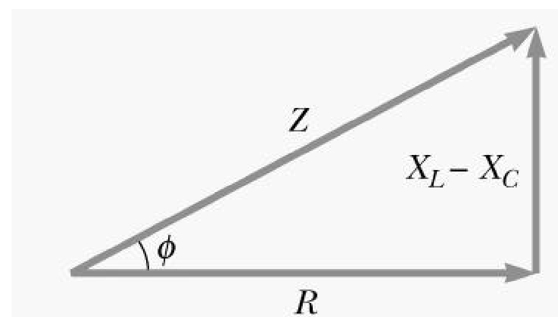
$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{V_{max}}{R + jX} = \frac{V_{max}}{\sqrt{R^2 + X^2}} e^{-jf} \qquad f = \tan^{-1}\frac{X}{R}$$

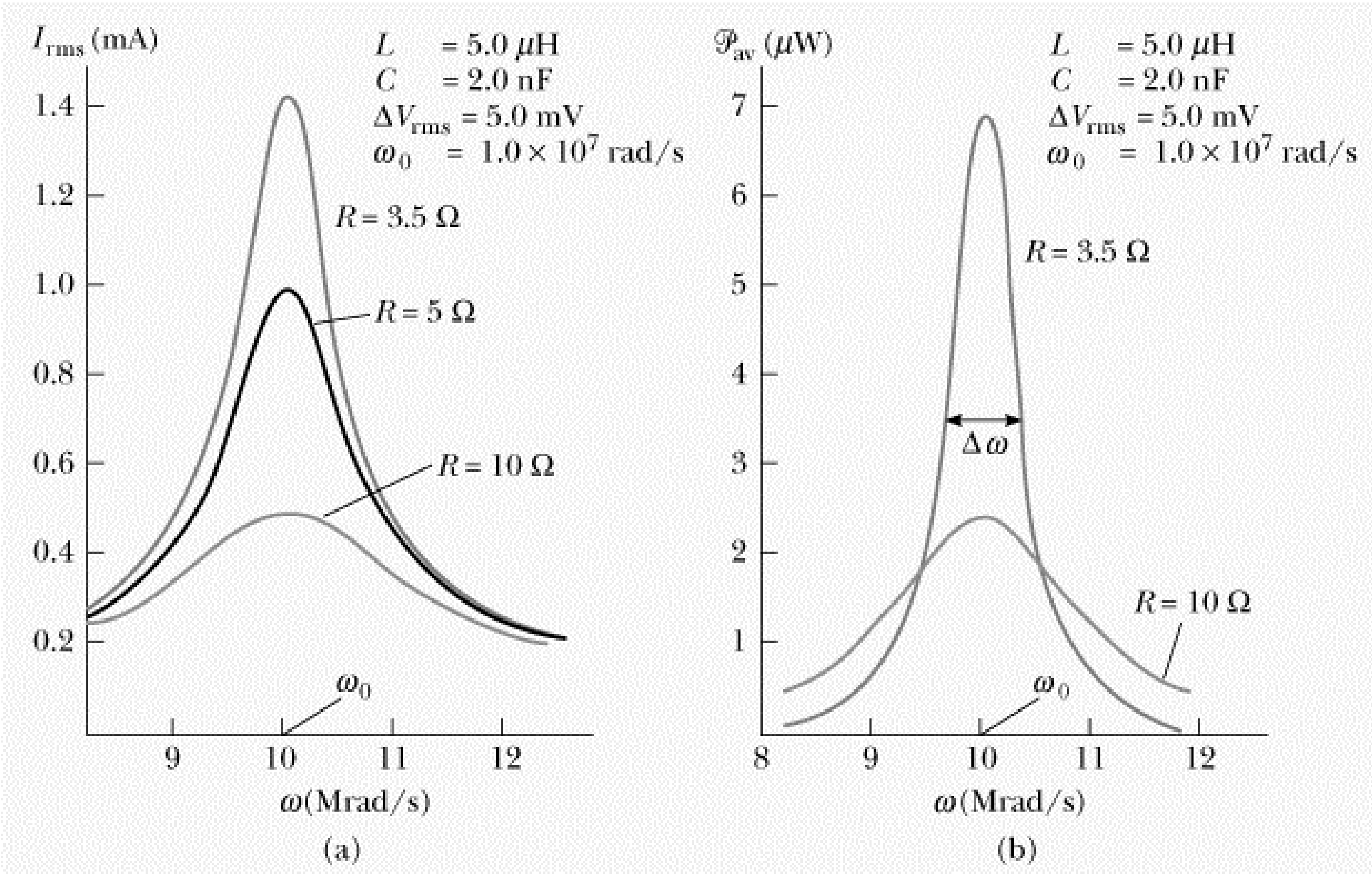


$$i(t) = \mathrm{Re}[e^{jwt}\mathbf{I}] = \frac{V_{max}}{\sqrt{R^2 + X^2}}\cos(wt - f) = I_0(w)\cos(wt - f)$$

$$I_0 = \frac{V_{max}}{\sqrt{R^2 + X^2}} = \frac{V_{max}}{\sqrt{R^2 + (wL - \frac{1}{wC})^2}}$$

$$I_{0\,max} = \frac{V_{max}}{R}$$



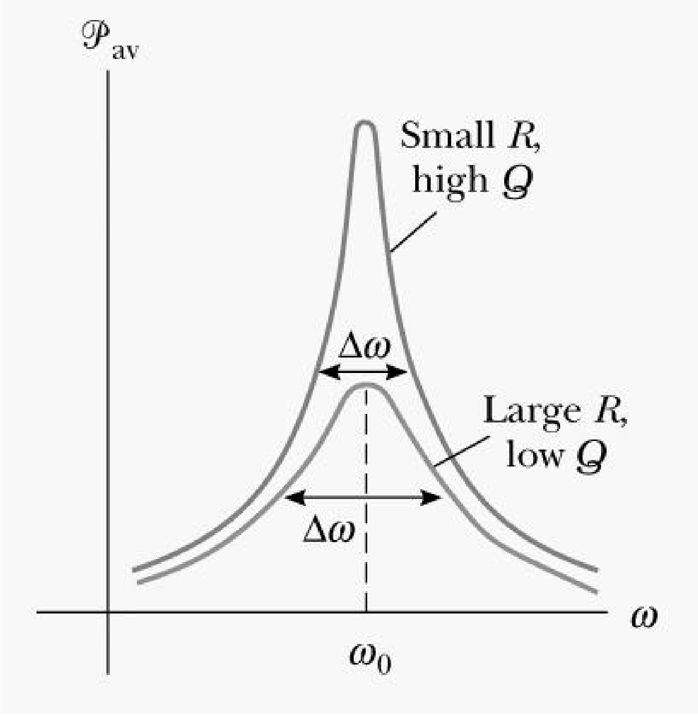

I_{rms} (mA)
L = 5.0 μH
C = 2.0 nF
ΔV_{rms} = 5.0 mV
ω_0 = 1.0×10^7 rad/s
1.4
1.2
1.0
0.8
0.6
0.4
0.2
R = 3.5 Ω
R = 5 Ω
R = 10 Ω
ω_0
9
10
11
12
ω(Mrad/s)
(a)
$\mathcal{P}_{av}$ (μW)
L = 5.0 μH
C = 2.0 nF
ΔV_{rms} = 5.0 mV
ω_0 = 1.0×10^7 rad/s
7
6
5
4
3
2
1
R = 3.5 Ω
Δω
R = 10 Ω
ω_0
8
9
10
11
12
ω(Mrad/s)
(b)

電路消耗的功率

$$P_{av} = I_{rms}^2 |Z| \cos f = I_{rms}^2 R = \frac{V_{rms}^2 R}{R^2 + (wL - \frac{1}{wC})^2}$$

$$(wL - \frac{1}{wC})^2 = \frac{L^2}{w^2}(w^2 - w_0^2)^2 \qquad w_0 = \frac{1}{\sqrt{LC}}$$

$$P_{av} = \frac{V_{rms}^2 R}{R^2 + (wL - \frac{1}{wC})^2} = \frac{V_{rms}^2 R w^2}{R^2 w^2 + L^2 (w^2 - w_0^2)^2}$$



當 $w = w_0$ $\quad P_{av} = \frac{V_{rms}^2}{R}$ **為最大值**

半功率點之間的寬度(width at the half-power points)**為** $\Delta w = \frac{R}{L}$

品質因子(quality factor) $Q \equiv \frac{w_0}{\Delta w}$ $\qquad Q = \frac{w_0 L}{R}$

平均儲存在系統的能量為

$$E=\frac{1}{2}CV_{\mathrm{Crms}}^{2}+\frac{1}{2}LI_{\mathrm{rms}}^{2}=\frac{1}{2}I_{\mathrm{rms}}^{2}(C\cdot\frac{1}{w^{2}C^{2}}+L)=\frac{1}{2}I_{\mathrm{rms}}^{2}L(\frac{1}{w^{2}LC}+1)$$

平均每週期消耗在系統的能量為

$$\Delta E=T\mathrm{P}_{\mathrm{av}}=\frac{2p}{w}I_{\mathrm{rms}}^{2}R$$

$$\frac{2pE}{\Delta E}=\frac{1}{2}\frac{wL}{R}(\frac{1}{w^{2}LC}+1)$$

在共振頻率時

$$\frac{2pE}{\Delta E}=\frac{w_{0}L}{R}=\mathrm{Q}=\frac{1}{\sqrt{LC}}\frac{L}{R}=\frac{1}{R}\sqrt{\frac{L}{C}}$$

品質因子又可看為在共振時

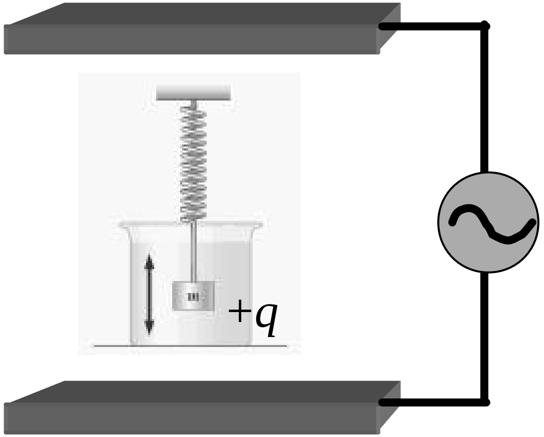
$$\mathrm{Q}=\frac{2\pi\ \text{平均儲存在系統的能量}}{\text{平均每週期消耗在系統的能量}}$$

品質因子愈高的振盪子，消耗的能量愈少，且頻譜響應之頻寬愈窄。

RLC電路與受迫振盪子的類比

考慮右圖中一阻尼振盪子受到週期性外力影響

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=f_0\cos wt$$



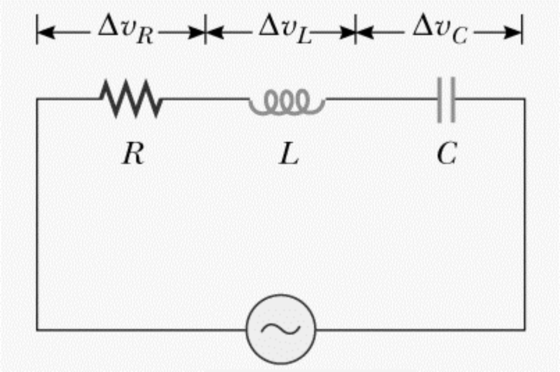
和RLC**電路的情形相比，非常類似**

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C}=V_0\cos wt$$

用相子方法來解穩態(steady state)**的解。**

$$-Lw^2\mathbf{Q}+jwR\mathbf{Q}+\frac{\mathbf{Q}}{C}=V_0$$

$$\mathbf{Q}=\frac{V_0}{-Lw^2+jwR+\frac{1}{C}}=\frac{V_0}{L(w_0^2-w^2)+jwR}$$



$V_0\cos wt$

Analogies Between Electrical and Mechanical Systems

Electric Circuit		One-Dimensional Mechanical System
Charge	$Q \leftrightarrow x$	Displacement
Current	$I \leftrightarrow v_x$	Velocity
Potential difference	$\Delta V \leftrightarrow F_x$	Force
Resistance	$R \leftrightarrow b$	Viscous damping coefficient
Capacitance	$C \leftrightarrow 1/k$	(k = spring constant)
Inductance	$L \leftrightarrow m$	Mass
Current = time derivative of charge	$I = \frac{dQ}{dt} \leftrightarrow v_x = \frac{dx}{dt}$	Velocity = time derivative of position
Rate of change of current = second time derivative of charge	$\frac{dI}{dt} = \frac{d^2Q}{dt^2} \leftrightarrow a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$	Acceleration = second time derivative of position
Energy in inductor	$U_L = \frac{1}{2}LI^2 \leftrightarrow K = \frac{1}{2}mv^2$	Kinetic energy of moving mass
Energy in capacitor	$U_C = \frac{1}{2}\frac{Q^2}{C} \leftrightarrow U = \frac{1}{2}kx^2$	Potential energy stored in a spring
Rate of energy loss due to resistance	$I^2R \leftrightarrow bv^2$	Rate of energy loss due to friction
RLC circuit	$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0 \leftrightarrow m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$	Damped mass on a spring

受迫簡諧振子的解

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=f_0\cos wt$$

$$-mw^2\mathbf{x}+jwb\mathbf{x}+k\mathbf{x}=f_0 \qquad w_0^2=k/m$$

$$\mathbf{x}=\frac{f_0}{-mw^2+jwb+k}=\frac{f_0/m}{(w_0^2-w^2)+jwb/m}$$

$$\mathrm{Q}=w_0\frac{m}{b}=\frac{w_0}{g} \qquad b=gm$$

受迫振盪常用來作為原子或材料中電子受到電磁輻射或光刺激產生能階躍遷之模型。